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Journal of Geometry and Physics 57 (2007) 1601–1630

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Supersymmetric harmonic maps into symmetric spaces

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Received 29 November 2005; received in revised form 12 January 2007; accepted 26 January 2007
Available online 29 January 2007

Abstract

We study supersymmetric harmonic maps from the point of view of integrable systems. We show that the superharmonic maps from $\mathbb{R}^{2|2}$ into a symmetric space are solutions of an integrable system and that we have a Weierstrass-type representation in terms of holomorphic potentials (as well as of meromorphic potentials). At the end of the paper we show that superprimitive maps from $\mathbb{R}^{2|2}$ into a 4-symmetric space give us, by restriction to $\mathbb{R}^2$, solutions of the second elliptic system associated with the previous 4-symmetric space.

Keywords: Superspace $\mathbb{R}^{2|2}$; Supersymmetric maps; Component fields; Supersymmetric Lagrangian; Harmonic maps; Integrable systems; Loop groups; Weierstrass representation of harmonic maps; Second elliptic system associated with a 4-symmetric space

0. Introduction

In this paper we study supersymmetric harmonic maps from the point of view of integrable systems. It is well known that harmonic maps from $\mathbb{R}^2$ into a symmetric space are solutions of an integrable system (see [8,4,3,12,13]). We show here that the superharmonic maps from $\mathbb{R}^{2|2}$ into a symmetric space are solutions of an integrable system, more precisely of a first elliptic integrable system in the sense of Terng (see [25]), and that we have a Weierstrass-type representation in terms of holomorphic potentials (as well as of meromorphic potentials). At the end of the paper we show that superprimitive maps from $\mathbb{R}^{2|2}$ into a 4-symmetric space give us, by restriction to $\mathbb{R}^2$, solutions of the second elliptic system associated with the previous 4-symmetric space. This leads us to conjecture that any second elliptic system associated with a 4-symmetric space has a geometrical interpretation in terms of surfaces with values in a symmetric space (such that a certain associated map is harmonic), as this is the case for Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces (see [17]) or for ρ-harmonic surfaces of $\mathbb{O}$ (see [19]).

In theoretical physics, harmonic maps between surfaces and Lie groups are extensively studied, since they lead to properties which are strongly analogous to (anti)self-dual Yang–Mills connections on four-dimensional manifolds, but they are simpler to handle. In such a context they correspond to the so-called σ-models, and in the presence of supersymmetries, to the supersymmetric σ-models (see [5]). Recently, the interest of physicists in these objects has been reinforced since their quantization leads to examples of conformal quantum field theories — an extremely

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.01.005

rich subject. In some sense the quantum theory for (super)harmonic maps between a (super)surface and an Einstein manifold corresponds to string theory. Moreover supersymmetry, which intermingles bosons and fermions, is essential to any good quantum field theory. All this constitutes a motivation to study superharmonic maps. Another motivation is that it is interesting to see how an integrable system is modified if we consider it in the supersymmetric context (when it is possible). Is it still an integrable system? And in this case, how is the algorithm which allows one to construct all solutions of this integrable system modified? In particular, what happens when we try to extend the Dorfmeister–Pedit–Wu method to the supersymmetric setting? The Dorfmeister–Pedit–Wu method was developed by J. Dorfmeister, F. Pedit and H.-Y. Wu for harmonic maps with values in a symmetric space (see [8]). This method allows one with the help of loop groups (see [23]) to obtain a Weierstrass-type representation in terms of holomorphic data for harmonic maps into symmetric spaces (see [8]), but also for Willmore surfaces (see [14]) and Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces (see [17]).

Thus, motivated by these questions, we study superharmonic maps from $\mathbb{R}^{2|2}$ into a symmetric space. Here, right from the start, the technical and conceptual difficulties are large. While physicists have an efficient calculus for computation in the supersymmetric setting, doing rigorous mathematics with this calculus takes one quite far afield: a map of supermanifolds is not a morphism of (ringed) topological spaces so much as a functor from the category of supermanifolds. So we have to find an efficient path through these problems. Moreover we have to prove foundational results in supergeometry (integration of Maurer–Cartan solutions; solvability of a $\bar{\partial}$-problem) to reach our conclusion. The resulting theory is more than a generalization of the classical theory: new phenomena arise of which the most striking is that the spectral parameter appears to second order in both Maurer–Cartan equations and the holomorphic data of the Dorfmeister–Pedit–Wu method. This is in sharp contrast to the classical theory and allows us to obtain a kind of supersymmetric interpretation of all the second elliptic systems associated with a 4-symmetric space.

Besides, we would like to mention related work by O'Dea in [22] which consists of a supersymmetric generalization of the work of K. Uhlenbeck. The focus of his work is on the uniton solutions constructed using a parametrized Lax pair by K. Uhlenbeck in 1989 (see [26]). She showed that harmonic maps into a unitary group can be factorized into a product of these simple maps.

Our paper is organized as follows. In the first section, we define superfields $\Phi\colon \mathbb{R}^{2|2} \to M$ from $\mathbb{R}^{2|2}$ to a Riemannian manifold, and component fields. Then we recall the functor of points approach to supermanifolds; we define the writing of a superfield and study its behaviour when we embed the manifold M in a Euclidean space $\mathbb{R}^N$. Lastly, we recall the derivation on $\mathbb{R}^{2|2}$. In Section 2 we introduce the supersymmetric Lagrangian on $\mathbb{R}^{2|2}$, define the supersymmetric maps and derive the Euler–Lagrange equations in terms of the component fields. Next, we study the case $M = S^n$: we write the Euler–Lagrange equations in this case and we derive from them the superharmonic map equation in this case. Then we introduce the superspace formulation of the Lagrangian and derive the superharmonic map equation for the general case of a Riemannian manifold M. In Section 3, we introduce the lift of a superfield with values in a symmetric space; then we express the superharmonic map equation in terms of the Maurer–Cartan form of the lift. Once more, in order to make comprehension easier, we first treat the case $M = S^n$, before the general case. In Section 4, we study the zero-curvature equation (i.e. the Maurer–Cartan equation) for a 1-form on $\mathbb{R}^{2|2}$ with values in a Lie algebra. This allows us to formulate the superharmonic map equation as the zero-curvature equation for a 1-form on $\mathbb{R}^{2|2}$ with values in a loop space $\Lambda\mathfrak{g}_\tau$. Then we make precise the extended Maurer–Cartan form, and characterize the superharmonic maps in terms of extended lifts. Section 5 deals with the Weierstrass representation: we define holomorphic functions and 1-forms in $\mathbb{R}^{2|2}$, and then we define holomorphic potentials. We show that we have a Weierstrass-type representation of the superharmonic maps in terms of holomorphic potentials. Lastly, we deal with meromorphic potentials. In Section 6, we make precise the Weierstrass representation in terms of the component fields. In Section 7, we study the superprimitive maps with values in 4-symmetric spaces, and we make precise their Weierstrass representation. This allows us in the last section to show that the restrictions to $\mathbb{R}^2$ of superprimitive maps are solutions of a second elliptic integrable system in the even part of a super-Lie algebra.

1. Definitions and notation

We consider the superspace $\mathbb{R}^{2|2}$ with coordinates $(x, y, \theta_1, \theta_2)$; (x, y) are the even coordinates and (θ_1, θ_2) the odd coordinates. Let M be a Riemannian manifold. We will be interested in maps $\Phi\colon \mathbb{R}^{2|2} \to M$ (which are even), i.e. morphisms of sheaves of super $\mathbb{R}$-algebras from $\mathbb{R}^{2|2}$ to M (see [6,1,20,21]). We call these maps *superfields*. We write such a superfield as

$$\Phi = u + \theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F' \tag{1}$$

where u, ψ_1, ψ_2, F' are the component fields (see [7]). We view these as maps from $\mathbb{R}^2$ into a supermanifold: u is a map from $\mathbb{R}^2$ to M, ψ_1, ψ_2 are odd sections of $u^*(TM)$ and F' is an even section of $u^*(TM)$. So u, F' are even whereas ψ_1, ψ_2 are odd. The supermanifold of superfields Φ is isomorphic to the supermanifold of component fields $\{u, \psi_1, \psi_2, F'\}$ (see [7]). Besides, the component fields can be defined as the restriction to $\mathbb{R}^2$ of certain derivatives of Φ:

$$\begin{aligned} u &= i^*\Phi\colon \mathbb{R}^2 \to M \\ \psi_a &= i^* D_a\Phi\colon \mathbb{R}^2 \to u^*(\Pi TM) \\ F' &= i^*\left(-\frac{1}{2}\varepsilon^{ab} D_a D_b \Phi\right)\colon \mathbb{R}^2 \to u^*(TM), \end{aligned} \tag{2}$$

where $i\colon \mathbb{R}^2 \to \mathbb{R}^{2|2}$ is the natural inclusion, Π is the functor which reverses the parity, and the left-invariant vector fields D_a are defined below. This is the definition of the component fields used in [7]. We use another definition based on the morphism interpretation of superfields, which is equivalent to the previous one, given by (2). Moreover as in [7] we use the functor of points approach to supermanifolds (see [6]). If B is a supermanifold, then a B-point of $\mathbb{R}^{2|2}$ is a morphism $B \to \mathbb{R}^{2|2}$. It can be viewed as a family of points of $\mathbb{R}^{2|2}$ parametrized by B, i.e. a section of the projection $\mathbb{R}^{2|2} \times B \to B$. Then a map Φ from $\mathbb{R}^{2|2}$ to M is a functor from the category of supermanifolds, which with each B associates a map $\Phi_B\colon \mathbb{R}^{2|2}(B) \to M(B)$ from the set of B-points of $\mathbb{R}^{2|2}$ to the set $M(B)$ of B-points of M. For example, if we take $B = \mathbb{R}^{0|L}$, which is the topological space $\mathbb{R}^0$ endowed with the Grassmann algebra $B_L = \mathbb{R}[\eta_1, \dots, \eta_L]$ over $\mathbb{R}^L$, then a $\mathbb{R}^{0|L}$-point of $\mathbb{R}^{2|2}$ is in the form $(x, y, \theta_1, \theta_2)$ where $x, y \in B_L^0$, the even part of B_L, and $\theta_1, \theta_2 \in B_L^1$, the odd part of B_L. Hence the set of $\mathbb{R}^{0|L}$-points of $\mathbb{R}^{2|2}$ is $B_L^{2|2} := (B_L^0)^2 \times (B_L^1)^2$. Thus if we restrict ourselves to the category of supermanifolds $\mathbb{R}^{0|L}$, $L \in \mathbb{N}$, then a map $\Phi\colon \mathbb{R}^{2|2} \to M$ is a sequence (Φ_L), of G^∞ functions defined by Rogers [24], such that Φ_L is a G^∞ function from $B_L^{2|2}$ to the G^∞ supermanifold over B_L, $M(\mathbb{R}^{0|L})$, and such that $\Phi_{L'}{}_{|B_L^{2|2}} = \Phi_L$, if $L \leq L'$. Hence, in this case, if we suppose $M = \mathbb{R}^n$, we have $M(\mathbb{R}^{0|L}) = B_L^{n|0} = (B_L^0)^n$ and the writing of (1) is the z expansion of Φ_L (see [24]). Further, following [9], we can say equivalently that if we denote by $\mathcal{F}$ the infinite dimensional supermanifold of morphisms: $\mathbb{R}^{2|2} \to M$, then the functor defined by Φ is a functor $B \mapsto \mathrm{Hom}(B, \mathcal{F})$: to each B there corresponds a B-point of $\mathcal{F}$, i.e. a morphism $\Phi_B\colon \mathbb{R}^{2|2} \times B \to M$. This means that the map Φ is a functor which with each B associates a morphism of algebras $\Phi_B^*\colon C^\infty(M) \to C^\infty(\mathbb{R}^{2|2} \times B)$. In concrete terms, throughout the paper, when we say: "Let $\Phi\colon \mathbb{R}^{2|2} \to M$ be a map", one can consider that it means "Let B be a supermanifold and let $\Phi_B\colon \mathbb{R}^{2|2} \times B \to M$ be a morphism" (omitting the additional condition that $B \mapsto \Phi_B$ is functorial in B). B can be viewed as a "space of parameters", and Φ_B as a family of maps: $\mathbb{R}^{2|2} \to M$, parametrized by B. We will never mention B, though it is tacitly assumed to always be there. Moreover, when we speak about morphisms, these are even morphisms, i.e. which preserve the parity, that is to say morphisms of super $\mathbb{R}$-algebras. Thus as said above, a superfield is even. But we will also be led to consider odd maps $A\colon \mathbb{R}^{2|2} \to M$; these are maps which give morphisms that reverse the parity.

Let us now make precise the writing of (1) and give our definition of the component fields.

In the general case (M is not a Euclidean space $\mathbb{R}^N$) the formal writing of (1) does not permit us to have directly the morphism of super-$\mathbb{R}$-algebras Φ^* as happens in the case $M = \mathbb{R}^N$, where the meaning of the writing of (1) is clear: it is the writing of the morphism Φ^*. Indeed, if $M = \mathbb{R}^N$ we have

$$\begin{aligned} &\forall f \in C^\infty(\mathbb{R}^N), \\ &\Phi^*(f) = f \circ \Phi = f(u) + \sum_{k=1}^{\infty} \frac{f^{(k)}(u)}{k!} \cdot (\theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F')^k \\ &\qquad = f(u) + \sum_{k=1}^{2} \frac{f^{(k)}(u)}{k!} \cdot (\theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F')^k \\ &\qquad = f(u) + \theta_1 \mathrm{d}f(u)\cdot\psi_1 + \theta_2 \mathrm{d}f(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{d}f(u)\cdot F' - \mathrm{d}^2 f(u)(\psi_1, \psi_2)) \end{aligned} \tag{3}$$

(we have used the fact that $\psi_1,\ \psi_2$ are odd). Then we define the component fields as the coefficient maps a_I in the decomposition $\Phi = \sum \theta^I a_I$ in the morphism writing, and as we will see below Eq. (2) follows from this definition.

In the general case, we must use local coordinates in M, to write the morphism of algebras Φ^* in the same way as (3) (see [1,20,21]). But the coefficient maps which appear in each chart in Eq. (3) written in each chart do not transform, through a change of chart, in such a way that they define some unique functions $u,\ \psi,\ F'$, which would allow us to give sense to (1) (in fact the coefficients corresponding to $u,\ \psi$ transform correctly but not the one corresponding to F'). So the writing of (1) does not have any sense if we do not make it precise. We will do it now. To do this we use the metric of M, more precisely its Levi-Civita connection (it was already used in the Eq. (2), taken in [7] as the definition of the component fields, where the outer (leftmost) derivative in the expression of F' is a covariant derivative). We will show that for any $\Phi\colon \mathbb{R}^{2|2} \to M$ there exist u, ψ, F' which satisfy the hypothesis above (u, F' even, ψ odd and ψ, F' are tangent) such that

$$\begin{aligned}&\forall f \in C^\infty(M),\\ &\quad \Phi^*(f) = f(u) + \theta_1 \mathrm{d}f(u)\cdot\psi_1 + \theta_2 \mathrm{d}f(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{d}f(u)\cdot F' - (\nabla \mathrm{d}f)(u)(\psi_1,\psi_2))\end{aligned} \tag{4}$$

where $\nabla \mathrm{d}f$ is the covariant derivative of $\mathrm{d}f$ (i.e. the covariant Hessian of f): $(\nabla \mathrm{d}f)(X,Y) = \langle \nabla_X(\nabla f), Y\rangle = \langle X, \nabla_Y(\nabla f)\rangle$. First, we remark that if (4) is true, then u, ψ, F' are unique. Then we can define the component fields as being u, ψ, F'; and (1) have a sense: it means that the morphism Φ^* is given by (4).

Now, to prove (4), let us embed isometrically M in a Euclidean space $\mathbb{R}^N$. Suppose first that M is defined by an implicit equation in $\mathbb{R}^N$: $f(x) = 0$, with $f\colon \mathbb{R}^N \to \mathbb{R}^{N-n}$ ($n = \dim M$). Then we have an isomorphism between $\{\text{superfields}\, \Phi\colon \mathbb{R}^{2|2} \to M\}$ and $\{\text{superfields}\, \Phi'\colon \mathbb{R}^{2|2} \to \mathbb{R}^N \mid \Phi'^*(f) = 0\}$; the isomorphism is

$$\Phi \longmapsto \Phi' = j\circ\Phi = (g \in C^\infty(\mathbb{R}^N) \mapsto \Phi^*(g_{|M})) \tag{5}$$

where $j\colon M \to \mathbb{R}^N$ is the natural inclusion. In particular, a superfield $\Phi'\colon \mathbb{R}^{2|2} \to \mathbb{R}^N$ is a superfield Φ from $\mathbb{R}^{2|2}$ into M if and only if $\Phi'^*(f) = f\circ\Phi' = 0$. This means that if we write $\Phi' = u + \theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F$ then we have by (3)

$$0 = f(u) + \theta_1 \mathrm{d}f(u)\cdot\psi_1 + \theta_2 \mathrm{d}f(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{d}f(u)\cdot F - \mathrm{d}^2 f(u)(\psi_1,\psi_2))$$

and hence $f(u) = 0,\ \mathrm{d}f(u)\cdot\psi_a = 0,\ \mathrm{d}f(u)\cdot F = \mathrm{d}^2 f(u)(\psi_1,\psi_2)$, i.e.

$$\begin{cases} u \text{ takes values in } M\\ \psi_a \text{ takes values in } u^*(TM)\\ \mathrm{d}f(u)\cdot F = \mathrm{d}^2 f(u)(\psi_1,\psi_2).\end{cases} \tag{6}$$

Thus a superfield $\Phi'\colon \mathbb{R}^{2|2} \to \mathbb{R}^N$ is "with values" in M if and only if $\Phi' = u + \theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F$ with (u,ψ,F) satisfying (6).

In the general case, there exists a family (U_α) of open sets in $\mathbb{R}^N$ such that $M \subset \bigcup_\alpha U_\alpha$ and C^∞ functions $f_\alpha\colon U_\alpha \to \mathbb{R}^{N-n}$ such that $M \cap U_\alpha = f_\alpha^{-1}(0)$. Then $\Phi \mapsto j\circ\Phi$ is an isomorphism between $\{\Phi\colon \mathbb{R}^{2|2} \to M\}$ and $\{\Phi'\colon \mathbb{R}^{2|2} \to \mathbb{R}^N \mid \Phi'^*(f_\alpha) = 0,\ \forall\alpha\}$. When we write $\Phi'^*(f_\alpha) = 0$, it means that we consider $V_\alpha = \Phi'^{-1}(U_\alpha)$ (it is the open submanifold of $\mathbb{R}^{2|2}$ associated with $u^{-1}(U_\alpha) \subset \mathbb{R}^2$, i.e. $u^{-1}(U_\alpha)$ endowed with the restriction to $u^{-1}(U_\alpha)$ of the structural sheaf of $\mathbb{R}^{2|2}$) and that $(\Phi'_{|V_\alpha})^*(f_\alpha) = f_\alpha\circ\Phi'_{|V_\alpha} = 0$ (see [6]). Hence a superfield $\Phi'\colon \mathbb{R}^{2|2} \to \mathbb{R}^N$ is with values in M if and only if $\Phi' = u + \theta_1\psi_1 + \theta_2\psi_2 + \theta_1\theta_2 F$ with (u,ψ,F) satisfying (6) for each f_α. Now, we write that we have $\Phi^*(g_{|M}) = \Phi'^*(g), \forall g \in C^\infty(\mathbb{R}^N)$:

$$\Phi^*(g_{|M}) = g(u) + \theta_1 \mathrm{d}g(u)\cdot\psi_1 + \theta_2 \mathrm{d}g(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{d}g(u)\cdot F - \mathrm{d}^2 g(u)(\psi_1,\psi_2)).$$

Let $\mathrm{pr}(x)\colon \mathbb{R}^N \to T_xM$ be the orthogonal projection on T_xM for $x \in M$, and $\mathrm{pr}^\perp(x) = Id - \mathrm{pr}(x)$; then set $F' = \mathrm{pr}(u)\cdot F$, $F^\perp = \mathrm{pr}^\perp(u)\cdot F$, so that $F = F' + F^\perp$. Let also $(e_1,\dots,e_{N-n})$ be a local moving frame of $TM^\perp$. Then we have

$$\mathrm{d}g(u)\cdot F - \mathrm{d}^2 g(u)(\psi_1,\psi_2) = \langle \nabla(g_{|M})(u), F'\rangle + \langle \nabla g(u), F^\perp\rangle - \langle D_{\psi_1}\nabla g(u), \psi_2\rangle$$

(where $D_{\psi_1} = \iota(\psi_1)\mathrm{d}$). Now using that ψ_1, ψ_2 are tangent to M at u,

$$\begin{aligned}\langle D_{\psi_1}\nabla g(u), \psi_2\rangle &= \langle \mathrm{pr}(u)\cdot(D_{\psi_1}\nabla g(u)), \psi_2\rangle\\ &= \langle \mathrm{pr}(u)\cdot[D_{\psi_1}(\mathrm{pr}()\cdot\nabla g)(u) + D_{\psi_1}(\mathrm{pr}^{\perp}()\cdot\nabla g)(u)], \psi_2\rangle\\ &= \langle \nabla_{\psi_1}\nabla(g_{|M}), \psi_2\rangle + \left\langle \mathrm{pr}(u)\cdot\left(D_{\psi_1}\sum_{i=1}^{N-n}\langle\nabla g, e_i\rangle e_i\right), \psi_2\right\rangle\\ &= \nabla \mathrm{d}(g_{|M})(u)(\psi_1,\psi_2) + \sum_{i=1}^{N-n}\langle \nabla g(u), e_i\rangle\langle \mathrm{d}e_i(u)\cdot\psi_1, \psi_2\rangle\end{aligned}$$

and then

$$\begin{aligned}\mathrm{d}g(u)\cdot F - \mathrm{d}^2 g(u)(\psi_1,\psi_2) &= \mathrm{d}(g_{|M})(u)\cdot F' - \nabla\mathrm{d}(g_{|M})(u)(\psi_1,\psi_2)\\ &\quad + \left\langle \mathrm{pr}^{\perp}(u)\cdot\nabla g(u), F^{\perp} - \sum_{i=1}^{N-n}\langle \mathrm{d}e_i(u)\cdot\psi_1,\psi_2\rangle e_i\right\rangle.\end{aligned}$$

But, as $\Phi^*(g_{|M})$ depends only on $h = g_{|M} \in C^\infty(M)$, we have

$$F^{\perp} = \sum_{i=1}^{N-n}\langle \mathrm{d}e_i(u)\cdot\psi_1,\psi_2\rangle e_i \tag{7}$$

and finally we obtain

$$\begin{aligned}&\forall h \in C^\infty(M),\\ &\quad \Phi^*(h) = h(u) + \theta_1 \mathrm{d}h(u)\cdot\psi_1 + \theta_2\mathrm{d}h(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{d}h(u)\cdot F' - (\nabla\mathrm{d}h)(u)(\psi_1,\psi_2))\end{aligned} \tag{8}$$

which is (4). And we have remarked that the coefficient maps $\{u, \psi, F'\}$ are unique, so in particular they do not depend on the embedding $M \hookrightarrow \mathbb{R}^N$. So we can define the multiplet of the component fields of Φ in the general case: it is the multiplet $\{u, \psi, F'\}$ which is defined by (4). It is an intrinsic definition. The isomorphism (5) leads to an isomorphism between the component fields

$$\{u, \psi, F'\} \longmapsto \{u, \psi, F\}.$$

The only change is in the third component field. We have $F' = \mathrm{pr}(u)\cdot F$, and the orthogonal component $F^{\perp}$ of F can be expressed in terms of (u, ψ) as we can see it for (7) or for (6).

In the following when we consider a manifold M with a natural embedding $M \hookrightarrow \mathbb{R}^N$, we will identify Φ and Φ', and we will talk about the two writings of Φ: its writing in M and its writing in $\mathbb{R}^N$. But when we refer to the component fields it will be always in M: $\{u, \psi, F'\}$. We will in fact use only the writing in $\mathbb{R}^N$ because it is more convenient to do computations, for example computations of derivatives or multiplication of two superfields with values in a Lie group, and because the meaning of the writing (1) in $\mathbb{R}^N$ is clear and well known as well as how to use it to do computations. So we will not use the writing in M. Our aim was, first, to show that it is possible to generalize the writing of (1) in the general case of a Riemannian manifold, then to give a definition of the component fields which did not use the derivatives of Φ (as in (2)), and above all to show how to deduce the component fields of Φ from its writing in $\mathbb{R}^N$: u, ψ are the same and $F' = \mathrm{pr}(u)\cdot F$.

Example 1. $M = S^n \subset \mathbb{R}^{n+1}$.

A superfield $\Phi\colon \mathbb{R}^{2|2} \to \mathbb{R}^{n+1}$ is a superfield $\Phi\colon \mathbb{R}^{2|2} \to S^n$ if and only if $\Phi^*(|\cdot|^2 - 1) = (|\cdot|^2 - 1)\circ\Phi = 0$ ($|\cdot|$ being the Euclidean norm in $\mathbb{R}^{n+1}$). This means that

$$0 = \langle\Phi,\Phi\rangle - 1 = |u|^2 - 1 + 2\theta_1\langle\psi_1, u\rangle + 2\theta_2\langle\psi_2, u\rangle + 2\theta_1\theta_2(\langle F, u\rangle - \langle\psi_1,\psi_2\rangle).$$

Thus $\Phi\colon \mathbb{R}^{2|2} \to \mathbb{R}^{n+1}$ takes values in S^n if and only if

$$\begin{cases} u \text{ takes values in } S^n\\ \psi_a \text{ is tangent to } S^n \text{ at } u\\ \langle F, u\rangle = \langle\psi_1,\psi_2\rangle.\end{cases}$$

In particular, in the case of S^n we have

$$F^{\perp} = \langle \psi_1, \psi_2 \rangle u.$$

1.1. Derivation on $\mathbb{R}^{2|2}$

Let us introduce the left-invariant vector fields of $\mathbb{R}^{2|2}$:

$$\begin{aligned} D_1 &= \frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial x} - \theta_2 \frac{\partial}{\partial y} \\ D_2 &= \frac{\partial}{\partial \theta_2} - \theta_1 \frac{\partial}{\partial y} + \theta_2 \frac{\partial}{\partial x}. \end{aligned}$$

These vectors fields induce odd derivations acting on superfields $D_a \Phi = \iota(D_a)\mathrm{d}\Phi$. Consider the case of superfields with values in $\mathbb{R}^N$. Write $\Phi = u + \theta_1 \psi_1 + \theta_2 \psi_2 + \theta_1 \theta_2 F$, a superfield $\Phi \colon \mathbb{R}^{2|2} \to \mathbb{R}^N$. Then we have

$$D_1 \Phi = \psi_1 - \theta_1 \frac{\partial u}{\partial x} + \theta_2 \left(F - \frac{\partial u}{\partial y} \right) + \theta_1 \theta_2 (\not{D}\psi)_1 \tag{9}$$

$$D_2 \Phi = \psi_2 - \theta_1 \left(\frac{\partial u}{\partial y} + F \right) + \theta_2 \frac{\partial u}{\partial x} + \theta_1 \theta_2 (\not{D}\psi)_2 \tag{10}$$

where

$$\not{D}\psi = \begin{pmatrix} \dfrac{\partial \psi_1}{\partial y} - \dfrac{\partial \psi_2}{\partial x} \\ -\dfrac{\partial \psi_1}{\partial x} - \dfrac{\partial \psi_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Hence

$$\begin{aligned} D_1 D_1 \Phi &= -\frac{\partial \Phi}{\partial x}, \qquad D_1 D_2 \Phi = -R(\Phi) - \frac{\partial \Phi}{\partial y}, \\ D_2 D_1 \Phi &= R(\Phi) - \frac{\partial \Phi}{\partial y}, \qquad D_2 D_2 \Phi = \frac{\partial \Phi}{\partial x}, \end{aligned}$$

where

$$\begin{aligned} R(\Phi) &:= F + \theta_1 \left(\frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} \right) + \theta_2 \left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} \right) + \theta_1 \theta_2 (\triangle u) \\ &:= F - \theta_1 (\not{D}\psi)_1 - \theta_2 (\not{D}\psi)_2 + \theta_1 \theta_2 (\triangle u). \end{aligned} \tag{11}$$

Thus

$$\begin{aligned} &D_1 D_2 - D_2 D_1 = -2R, \qquad [D_1, D_2] = D_1 D_2 + D_2 D_1 = -2 \frac{\partial}{\partial y} \\ &[D_1, D_1] = 2 D_1^2 = -2 \frac{\partial}{\partial x}, \qquad [D_2, D_2] = 2 \frac{\partial}{\partial x} \end{aligned}$$

(Throughout the paper, we denote by [,] the superbracket in the considered super-Lie algebra.)

Let us set

$$\begin{aligned} D &= \frac{1}{2}(D_1 - i D_2) = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \\ \bar{D} &= \frac{1}{2}(D_1 + i D_2) = \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \bar{z}} \end{aligned}$$

where $\theta = \theta_1 + i\theta_2$, $\frac{\partial}{\partial \theta} = \frac{1}{2}(\frac{\partial}{\partial \theta_1} - i \frac{\partial}{\partial \theta_2})$. Setting $\psi = \psi_1 - i\psi_2$, we can write $\Phi = u + \frac{1}{2}(\theta \psi + \bar{\theta} \bar{\psi}) + \frac{i}{2} \theta \bar{\theta} F$, and thus

$$D\Phi = \frac{1}{2}\psi - \theta\frac{\partial u}{\partial z} + \frac{i}{2}\bar{\theta}F - \frac{1}{2}\theta\bar{\theta}\frac{\partial\bar{\psi}}{\partial z} \tag{12}$$

$$\bar{D}\Phi = \frac{1}{2}\bar{\psi} - \bar{\theta}\frac{\partial u}{\partial\bar{z}} - \frac{i}{2}\theta F + \frac{1}{2}\theta\bar{\theta}\frac{\partial\psi}{\partial\bar{z}}. \tag{13}$$

Then

$$\begin{aligned} D\bar{D} &= \frac{1}{4}(D_1 - iD_2)(D_1 + iD_2) = \frac{1}{4}(D_1^2 + D_2^2 + i(D_1D_2 - D_2D_1)) \\ &= \frac{i}{4}(D_1D_2 - D_2D_1) = -\frac{i}{2}R \end{aligned}$$

and hence

$$D\bar{D} = -\bar{D}D = -\frac{i}{2}R.$$

We have also $D^2 = -\frac{\partial}{\partial z}$, $\bar{D}^2 = -\frac{\partial}{\partial\bar{z}}$. Let us compute $\bar{D}D\Phi$:

$$\begin{aligned} \bar{D}D\Phi &= \bar{D}\left(\frac{1}{2}\psi - \theta\frac{\partial u}{\partial z} + \frac{i}{2}\bar{\theta}F - \frac{1}{2}\theta\bar{\theta}\frac{\partial\bar{\psi}}{\partial z}\right) \\ &= \frac{i}{2}F + \frac{\theta}{2}\frac{\partial\bar{\psi}}{\partial z} - \frac{\bar{\theta}}{2}\frac{\partial\psi}{\partial\bar{z}} - \theta\bar{\theta}\frac{\partial}{\partial\bar{z}}\left(\frac{\partial u}{\partial z}\right) \\ &= \frac{i}{2}F + i\,\mathrm{Im}\left(\theta\frac{\partial\bar{\psi}}{\partial z}\right) - \frac{\theta\bar{\theta}}{4}(\triangle u). \end{aligned} \tag{14}$$

Let us denote by $i\colon \mathbb{R}^2 \to \mathbb{R}^{2|2}$ the natural inclusion; then using (9)–(11) we have

$$\begin{aligned} u &= i^*\Phi \\ \psi_a &= i^*D_a\Phi \\ F &= i^*\left(-\frac{1}{2}\varepsilon^{ab}D_aD_b\Phi\right) \end{aligned}$$

and we recover (2) for $M = \mathbb{R}^N$.

Let us return to the general case of superfields with values in M. In order to write (2) in M, we need a covariant derivative in the expression of F' to define the action of D_a on a section of the bundle Φ^*TM. In order to do this we use the pullback of the Levi-Civita connection. Suppose that M is isometrically embedded in $\mathbb{R}^N$. Let X be a section of Φ^*TM (for example $X = D_b\Phi$); then using the writing in $\mathbb{R}^N$ (i.e. considering that a map with values in M takes values in $\mathbb{R}^N$) we have

$$\nabla_{D_a}X = \mathrm{pr}(\Phi)\cdot D_aX.$$

Let us make precise the expression $\mathrm{pr}(\Phi)\cdot D_aX$. The projection pr is a map from M into $\mathcal{L}(\mathbb{R}^N)$, the algebra of endomorphisms of $\mathbb{R}^N$. We consider $\mathrm{pr}\circ\Phi$ which we write as $\mathrm{pr}(\Phi)$. Then considering the maps $\mathrm{pr}(\Phi)\colon \mathbb{R}^{2|2} \to \mathcal{L}(\mathbb{R}^N)$, $D_aX\colon \mathbb{R}^{2|2} \to \mathbb{R}^N$, and $B\colon (A, v) \in \mathcal{L}(\mathbb{R}^N) \times \mathbb{R}^N \mapsto A.v$, we form $B(\mathrm{pr}(\Phi), D_aX)\colon \mathbb{R}^{2|2} \to \mathbb{R}^N$. Now, since $\mathcal{L}(\mathbb{R}^N)$ is a finite dimensional vector space we can write from (4)

$$\mathrm{pr}(\Phi) = \Phi^*(\mathrm{pr}) = \mathrm{pr}(u) + \theta_1\mathrm{dpr}(u)\cdot\psi_1 + \theta_2\mathrm{dpr}(u)\cdot\psi_2 + \theta_1\theta_2(\mathrm{dpr}(u)\cdot F' - (\nabla\mathrm{dpr})(u)(\psi_1, \psi_2))$$

(we cannot use (3) because pr is only defined on M). This is the writing of the superfield $\mathrm{pr}\circ\Phi\colon \mathbb{R}^{2|2} \to \mathcal{L}(\mathbb{R}^N)$, so we can write

$$i^*(\nabla_{D_a}D_b\Phi) = i^*(\mathrm{pr}(\Phi)\cdot D_aD_b\Phi) = \mathrm{pr}(u)\cdot i^*(D_aD_b\Phi)$$

and thus $i^*(-\frac{1}{2}\varepsilon^{ab}\nabla_{D_a}D_b\Phi) = \mathrm{pr}(u)\cdot F = F'$. So we have (2) in the general case.

Example 2. $M = S^n \subset \mathbb{R}^{n+1}$.

We have $\mathrm{pr}(x) = \mathrm{Id} - \langle \cdot, x\rangle x$ for $x \in S^n$. So for X a section of $\Phi^* T S^n$, we have

$$\nabla_{D_a} X = D_a X - \langle D_a X, \Phi\rangle \Phi.$$

2. Supersymmetric Lagrangian

2.1. Euler–Lagrange equations

We consider the following supersymmetric Lagrangian (see [7]):

$$L = -\frac{1}{2}|\mathrm{d}u|^2 + \frac{1}{2}\langle \psi\ \not{D}_u \psi\rangle + \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle \psi_a, R(\psi_b, \psi_c)\psi_d\rangle + \frac{1}{2}|F'|^2 \tag{15}$$

where $\langle \psi\ \not{D}_u \psi\rangle = \langle \psi_1, (\not{D}_u\psi)_2\rangle - \langle \psi_2, (\not{D}_u\psi)_1\rangle$, R is the curvature of M and

$$\not{D}_u \psi = \begin{pmatrix} \dfrac{\partial \psi_1}{\partial y} - \dfrac{\partial \psi_2}{\partial x} \\ -\dfrac{\partial \psi_1}{\partial x} - \dfrac{\partial \psi_2}{\partial y} \end{pmatrix}$$

($\frac{\partial \psi_k}{\partial x_i}$ is of course a covariant derivative). This Lagrangian can be obtained by reduction to $\mathbb{R}^{2|2}$ of the supersymmetric σ-model Lagrangian on $\mathbb{R}^{3|2}$ (see [7]). We associate with this Lagrangian the action $\mathcal{A}(\Phi) = \int L(\Phi)\mathrm{d}x\mathrm{d}y$. It is a functional on the multiplets of components fields $\{u, \psi, F'\}$ of superfields $\Phi\colon \mathbb{R}^{2|2} \to M$, which is supersymmetric.

Definition 1. A superfield $\Phi\colon \mathbb{R}^{2|2} \to M$ is superharmonic if it is a critical point of the action $\mathcal{A}$.

Theorem 1. *If we suppose that $\nabla R = 0$ in M (the covariant derivative of the curvature vanishes) then the Euler–Lagrange equations associated with the action $\mathcal{A}$ are*

$$\boxed{\begin{aligned} \triangle u &= \frac{1}{2}(R(\psi_1, \psi_1) - R(\psi_2, \psi_2))\frac{\partial u}{\partial x} + R(\psi_1, \psi_2)\frac{\partial u}{\partial y} \\ \not{D}_u \psi &= \begin{pmatrix} R(\psi_1, \psi_2)\psi_1 \\ -R(\psi_1, \psi_2)\psi_2 \end{pmatrix} \\ F' &= 0 \end{aligned}} \tag{16}$$

Proof. We compute the variation of each term in the Lagrangian, keeping in mind that ψ_1, ψ_2 are odd (so their coordinates anticommute, $\psi_1^i \psi_2^j = -\psi_2^j \psi_1^i$):

-
$$\delta\left(\frac{1}{2}|\mathrm{d}u|^2\right) = \langle -\triangle u, \delta u\rangle + \mathrm{div}(\langle \mathrm{d}u, \delta u\rangle)$$

-
$$\begin{aligned} \delta\left(\frac{1}{2}\langle \psi\ \not{D}_u \psi\rangle\right) &= \frac{1}{2}(\langle \delta_\nabla \psi_1, (\not{D}_u\psi)_2\rangle + \langle \psi_1, \delta_\nabla(\not{D}_u\psi)_2\rangle - \langle \delta_\nabla \psi_2, (\not{D}_u\psi)_1\rangle - \langle \psi_2, \delta_\nabla(\not{D}_u\psi)_1\rangle) \\ &= \frac{1}{2}\Bigg[\langle \delta_\nabla \psi_1, (\not{D}_u\psi)_2\rangle - \langle \delta_\nabla \psi_2, (\not{D}_u\psi)_1\rangle + \left\langle \psi_1, -\frac{\partial}{\partial x}\delta_\nabla \psi_1 - \frac{\partial}{\partial y}\delta_\nabla \psi_2\right\rangle \\ &\quad - \left\langle \psi_2, \frac{\partial}{\partial y}\delta_\nabla \psi_1 - \frac{\partial}{\partial x}\delta_\nabla \psi_2\right\rangle + \left\langle \psi_1, R\left(\delta u, -\frac{\partial u}{\partial x}\right)\psi_1 - R\left(\delta u, -\frac{\partial u}{\partial y}\right)\psi_2\right\rangle \\ &\quad - \left\langle \psi_2, R\left(\delta u, \frac{\partial u}{\partial y}\right)\psi_1 + R\left(\delta u, -\frac{\partial u}{\partial x}\right)\psi_2\right\rangle\Bigg] \end{aligned}$$

where we have used $\delta_\nabla \frac{\partial \psi_k}{\partial x_i} - \frac{\partial}{\partial x_i}\delta_\nabla \psi_k = R(\delta u, \frac{\partial u}{\partial x_i})\psi_k$. Then we write that

$$\left\langle \psi_a, \frac{\partial}{\partial x_i}\delta_\nabla \psi_b \right\rangle = -\left\langle \frac{\partial \psi_a}{\partial x_i}, \delta_\nabla \psi_b \right\rangle + \frac{\partial}{\partial x_i}\langle \psi_a, \delta_\nabla \psi_b\rangle = \left\langle \delta_\nabla \psi_b, \frac{\partial \psi_a}{\partial x_i} \right\rangle + \frac{\partial}{\partial x_i}\langle \psi_a, \delta_\nabla \psi_b\rangle$$

and that

$$\left\langle \psi_a, R\left(\delta u, \frac{\partial u}{\partial x_i}\right)\psi_b \right\rangle = \left\langle R(\psi_b, \psi_a)\frac{\partial u}{\partial x_i}, \delta u \right\rangle$$

and thus we obtain

$$\begin{aligned}\delta\left(\frac{1}{2}\langle \psi \not{D}_u \psi\rangle\right) = \frac{1}{2}\Bigg[&\left\langle \delta_\nabla \psi_1, (\not{D}_u\psi)_2 + \left(-\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial y}\right)\right\rangle - \left\langle \delta_\nabla \psi_2, (\not{D}_u\psi)_1 + \left(\frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}\right)\right\rangle \\ &+ \frac{\partial}{\partial x}(-\langle \psi_1, \delta_\nabla \psi_1\rangle + \langle \psi_2, \delta_\nabla \psi_2\rangle) + \frac{\partial}{\partial y}(-\langle \psi_1, \delta_\nabla \psi_2\rangle - \langle \psi_2, \delta_\nabla \psi_1\rangle) \\ &- \left\langle \left(R(\psi_1, \psi_1)\frac{\partial u}{\partial x} + R(\psi_2, \psi_1)\frac{\partial u}{\partial y} + R(\psi_1, \psi_2)\frac{\partial u}{\partial y} - R(\psi_2, \psi_2)\frac{\partial u}{\partial x}\right), \delta u\right\rangle\Bigg]\end{aligned}$$

and finally

$$\begin{aligned}\delta\left(\frac{1}{2}\langle \psi \not{D}_u \psi\rangle\right) &= \langle \delta_\nabla \psi_1, (\not{D}_u\psi)_2\rangle - \langle \delta_\nabla \psi_2, (\not{D}_u\psi)_1\rangle \\ &- \left\langle \left[\frac{1}{2}(R(\psi_1, \psi_1) - R(\psi_2, \psi_2))\frac{\partial u}{\partial x} + R(\psi_1, \psi_2)\frac{\partial u}{\partial y}\right], \delta u\right\rangle + \operatorname{div}(\cdots)\end{aligned}$$

•

$$\begin{aligned}&\delta\left(\frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle \psi_a, R(\psi_b, \psi_c)\psi_d\rangle\right) \\ &= \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}(\nabla_{\delta u}R(\psi_b, \psi_c, \psi_d, \psi_a) + R(\delta\psi_a, \psi_b, \psi_c, \psi_d) + R(\psi_a, \delta\psi_b, \psi_c, \psi_d) \\ &\quad + R(\psi_a, \psi_b, \delta\psi_c, \psi_d) + R(\psi_a, \psi_b, \psi_c, \delta\psi_d)) \\ &= \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}(0 + \langle \delta\psi_a, R(\psi_b, \psi_c)\psi_d\rangle + \langle \delta\psi_b, R(\psi_d, \psi_a)\psi_c\rangle + \langle \delta\psi_c, R(\psi_d, \psi_a)\psi_b\rangle \\ &\quad + \langle \delta\psi_d, R(\psi_b, \psi_c)\psi_a\rangle) \quad \text{(using the symmetries of } R) \\ &= \frac{1}{12}(\langle \delta\psi_1, R(\psi_2, \psi_1)\psi_2 - R(\psi_2, \psi_2)\psi_1 - R(\psi_2, \psi_2)\psi_1 + R(\psi_1, \psi_2)\psi_2 + R(\psi_2, \psi_1)\psi_2 \\ &\quad - R(\psi_2, \psi_2)\psi_1 - R(\psi_2, \psi_2)\psi_1 + R(\psi_1, \psi_2)\psi_2\rangle + \langle \delta\psi_2, -R(\psi_1, \psi_1)\psi_2 + R(\psi_1, \psi_2)\psi_1 \\ &\quad + R(\psi_2, \psi_1)\psi_1 - R(\psi_1, \psi_1)\psi_2 - R(\psi_1, \psi_1)\psi_2 + R(\psi_1, \psi_2)\psi_1 + R(\psi_2, \psi_1)\psi_1 - R(\psi_1, \psi_1)\psi_2\rangle) \\ &= \frac{1}{12}(\langle \delta\psi_1, -4R(\psi_2, \psi_2)\psi_1 + 4R(\psi_1, \psi_2)\psi_2\rangle + \langle \delta\psi_2, 4R(\psi_2, \psi_1)\psi_1 - 4R(\psi_1, \psi_1)\psi_2\rangle) \\ &= \frac{1}{3}(\langle \delta\psi_1, R(\psi_1, \psi_2)\psi_2 - R(\psi_2, \psi_2)\psi_1\rangle + \langle \delta\psi_2, R(\psi_2, \psi_1)\psi_1 - R(\psi_1, \psi_1)\psi_2\rangle).\end{aligned}$$

Finally, by using the Bianchi identity we obtain

$$\delta\left(\frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle \psi_a, R(\psi_b, \psi_c)\psi_d\rangle\right) = \langle \delta_\nabla\psi_1, R(\psi_1, \psi_2)\psi_2\rangle + \langle \delta_\nabla\psi_2, R(\psi_2, \psi_1)\psi_1\rangle.$$

•

$$\delta\left(\frac{1}{2}|F'|^2\right) = \langle F', \delta_\nabla F'\rangle.$$

Hence the first variation of the Lagrangian is

$$\begin{aligned}\delta\mathcal{L} = \int \Bigg[&\left\langle \triangle u - \frac{1}{2}(R(\psi_1, \psi_1) - R(\psi_2, \psi_2))\frac{\partial u}{\partial x} - R(\psi_1, \psi_2)\frac{\partial u}{\partial y}, \delta u\right\rangle \\ &+ \langle \delta_\nabla \psi_1, (\not{D}_u\psi)_2 + R(\psi_1, \psi_2)\psi_2\rangle - \langle \delta_\nabla \psi_2, (\not{D}_u\psi)_1 - R(\psi_1, \psi_2)\psi_1\rangle + \langle F', \delta_\nabla F'\rangle\Bigg]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

This completes the proof of the theorem. ■

Remark 1. In any symmetric space, $\nabla R = 0$, so the preceding result holds. Moreover in the general case of a Riemannian manifold M the Euler–Lagrange equations are obtained by adding to the right hand side of the first equation of (16) the term $-\frac{1}{2}(\nabla_{\psi_1} R)(\psi_1, \psi_2)\psi_2$.

2.2. The case $M = S^n$

The curvature of S^n is given by

$$\begin{aligned} R(X, Y, Z, T) &= \langle X, T\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle T, Y\rangle \\ &= (\delta^{il}\delta^{jk} - \delta^{ik}\delta^{jl})X_i Y_j Z_k T_l \end{aligned}$$

so

$$\begin{aligned} R(V_1, V_2)V_3 &= \langle V_2, V_3\rangle V_1 + \langle V_1, V_3\rangle V_2 \\ R(V_1, V_2)Z &= -\langle V_2, Z\rangle V_1 - \langle V_1, Z\rangle V_2 \end{aligned}$$

where V_1, V_2, V_3 are odd and Z is even.

Thus the Euler–Lagrange equations for S^n are

$$\begin{aligned} &\triangle u + |\mathrm{d}u|^2 u = -\left\langle \psi_1, \frac{\partial u}{\partial x}\right\rangle \psi_1 + \left\langle \psi_2, \frac{\partial u}{\partial x}\right\rangle \psi_2 - \left(\left\langle \psi_2, \frac{\partial u}{\partial y}\right\rangle \psi_1 + \left\langle \psi_1, \frac{\partial u}{\partial y}\right\rangle \psi_2\right) \\ &\not{D}_u \psi = \begin{pmatrix} \langle \psi_2, \psi_1\rangle \psi_1 \\ \langle \psi_2, \psi_1\rangle \psi_2 \end{pmatrix} \\ &F = \langle \psi_1, \psi_2\rangle u. \end{aligned}$$

Let us now rewrite these equations by using the complex variable and setting $\psi = \psi_1 - i\psi_2$:

$$\boxed{\begin{aligned} 4\frac{\partial^\nabla}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) &= \left(\psi \left\langle \psi, \frac{\partial u}{\partial \bar{z}}\right\rangle + \bar{\psi}\left\langle \bar{\psi}, \frac{\partial u}{\partial z}\right\rangle\right) \\ \frac{\partial^\nabla \psi}{\partial \bar{z}} &= \frac{1}{4}\langle \bar{\psi}, \psi\rangle \bar{\psi} \\ F &= \frac{1}{2i}\langle \psi, \bar{\psi}\rangle u \end{aligned}}. \tag{17}$$

Theorem 2. *Let $\Phi : \mathbb{R}^{2|2} \to S^n$ be a superfield; then Φ is superharmonic if and only if*

$$\bar{D}D\Phi + \langle \bar{D}\Phi, D\Phi\rangle \Phi = 0 \tag{18}$$

in $\mathbb{R}^{n+1}$.

Proof. According to (14), we have

$$\bar{D}D\Phi = \frac{i}{2}F + i\,\mathrm{Im}\left(\theta \frac{\partial \bar{\psi}}{\partial z}\right) - \theta\bar{\theta}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right).$$

Moreover, by using (12) and (13)

$$\begin{aligned} \langle \bar{D}\Phi, D\Phi\rangle \Phi = &\frac{1}{4}\langle \bar{\psi}, \psi\rangle + \theta\left(\frac{1}{2}\left\langle \bar{\psi}, \frac{\partial u}{\partial z}\right\rangle - \frac{i}{4}\langle F, \psi\rangle\right) + \bar{\theta}\left(-\frac{1}{2}\left\langle \frac{\partial u}{\partial \bar{z}}, \psi\right\rangle - \frac{i}{4}\langle \bar{\psi}, F\rangle\right) \\ &+ \theta\bar{\theta}\left(-\frac{1}{4}\left\langle \bar{\psi}, \frac{\partial \bar{\psi}}{\partial z}\right\rangle + \frac{1}{4}\left\langle \frac{\partial \psi}{\partial \bar{z}}, \psi\right\rangle + \frac{1}{4}|F|^2 - \left\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial u}{\partial z}\right\rangle\right). \end{aligned}$$

But since $\langle\psi, u\rangle = \langle\bar{\psi}, u\rangle = 0$ we have $\langle\bar{\psi}, \frac{\partial u}{\partial z}\rangle = -\langle\frac{\partial\bar{\psi}}{\partial z}, u\rangle$ and $\langle\frac{\partial u}{\partial\bar{z}}, \psi\rangle = -\langle u, \frac{\partial\psi}{\partial\bar{z}}\rangle$ so

$$\begin{aligned}\langle\bar{D}\Phi, D\Phi\rangle\Phi &= \frac{1}{4}\langle\bar{\psi},\psi\rangle - \theta\left(\frac{1}{2}\left\langle\frac{\partial\bar{\psi}}{\partial z}, u\right\rangle + \frac{i}{4}\langle F,\psi\rangle\right) + \bar{\theta}\left(\frac{1}{2}\left\langle\frac{\partial\psi}{\partial\bar{z}}, u\right\rangle - \frac{i}{4}\langle\bar{\psi}, F\rangle\right)\\ &\quad + \theta\bar{\theta}\left(\frac{1}{2}\mathrm{Re}\left(\left\langle\frac{\partial\psi}{\partial\bar{z}},\psi\right\rangle\right) + \frac{1}{4}|F|^2 - \left\langle\frac{\partial u}{\partial\bar{z}}, \frac{\partial u}{\partial z}\right\rangle\right).\end{aligned}$$

Hence

$$\begin{aligned}\bar{D}D\Phi + \langle\bar{D}\Phi, D\Phi\rangle\Phi &= \bar{D}D\Phi + \langle\bar{D}\Phi, D\Phi\rangle\left(u + \frac{1}{2}(\theta\psi + \bar{\theta}\bar{\psi}) + \frac{i}{2}\theta\bar{\theta}F\right)\\ &= \left(\frac{i}{2}F + \frac{1}{4}\langle\bar{\psi},\psi\rangle u\right) + \frac{\theta}{2}\left(\frac{\partial\bar{\psi}}{\partial z} - \left\langle\frac{\partial\bar{\psi}}{\partial z}, u\right\rangle u + \frac{1}{4}\langle\bar{\psi},\psi\rangle\psi - \frac{i}{2}\langle F,\psi\rangle u\right)\\ &\quad + \frac{\bar{\theta}}{2}\left(-\frac{\partial\psi}{\partial\bar{z}} + \left\langle\frac{\partial\psi}{\partial\bar{z}}, u\right\rangle u + \frac{1}{4}\langle\bar{\psi},\psi\rangle\bar{\psi} - \frac{i}{2}\langle\bar{\psi}, F\rangle u\right)\\ &\quad + \theta\bar{\theta}\left(-\left[\frac{\partial}{\partial\bar{z}}\frac{\partial u}{\partial z} + \left\langle\frac{\partial u}{\partial\bar{z}}, \frac{\partial u}{\partial z}\right\rangle\right] + \frac{1}{4}\left[\psi\left\langle\psi, \frac{\partial u}{\partial\bar{z}}\right\rangle + \bar{\psi}\left\langle\bar{\psi}, \frac{\partial u}{\partial z}\right\rangle\right]\right.\\ &\quad \left. + \frac{i}{8}\langle F,\psi\rangle\bar{\psi} - \frac{i}{8}\langle\bar{\psi}, F\rangle\psi + \left[\frac{1}{4}|F|^2 + \frac{1}{2}\mathrm{Re}\left(\left\langle\frac{\partial\psi}{\partial\bar{z}},\psi\right\rangle\right)\right]u + \frac{i}{8}\langle\bar{\psi},\psi\rangle F\right).\end{aligned}$$

So we see that if Φ satisfies (17) then this expression vanishes because $\langle F,\psi\rangle = \langle F,\bar{\psi}\rangle = 0$ and $\mathrm{Re}\left(\left\langle\frac{\partial\psi}{\partial\bar{z}},\psi\right\rangle\right) = \mathrm{Re}\langle\bar{\psi},\psi\rangle^2 = -4|F|^2$ by using (17).

Conversely, if this expression vanishes then the vanishing of the first term gives us the third equation of (17), and thus we have $\langle F,\psi\rangle = 0$ and so the vanishing of the term in θ gives us the second equation of (17). Lastly the first equation of (17) is given by the vanishing of the term in $\theta\bar{\theta}$ and by using the second and third equations of (17). This completes the proof. ■

Remark 2. Eq. (18) is the analogue of the equation for harmonic maps $u\colon \mathbb{R}^2 \to S^n$:

$$\frac{\partial}{\partial\bar{z}}\left(\frac{\partial u}{\partial z}\right) + \left\langle\frac{\partial u}{\partial\bar{z}}, \frac{\partial u}{\partial z}\right\rangle = 0.$$

In fact, Eq. (18) means that

$$\nabla_{\bar{D}}D\Phi = 0.$$

Indeed we have $\nabla_{\bar{D}}D\Phi = \mathrm{pr}(\Phi)\cdot\bar{D}D\Phi = \bar{D}D\Phi - \langle\bar{D}D\Phi, \Phi\rangle\Phi$ but

$$\begin{aligned}\langle\bar{D}D\Phi, \Phi\rangle\Phi &= \bar{D}(\langle D\Phi, \Phi\rangle) + \langle D\Phi, \bar{D}\Phi\rangle\\ &= 0 - \langle\bar{D}\Phi, D\Phi\rangle\end{aligned}$$

because $\langle\Phi,\Phi\rangle = 1 \Longrightarrow \langle D\Phi, \Phi\rangle = 0$. So

$$\nabla_{\bar{D}}D\Phi = \bar{D}D\Phi + \langle\bar{D}\Phi, D\Phi\rangle\Phi.$$

It is a general result that $\Phi\colon \mathbb{R}^{2|2} \to M$ (Riemannian without other hypothesis) is superharmonic if and only if $\nabla_{\bar{D}}D\Phi = 0$. To prove it we need to use the superspace formulation for the supersymmetric Lagrangian. This is what we are going to do now.

2.3. The superspace formulation

We consider the Lagrangian density on $\mathbb{R}^{2|2}$ (see [7]):

$$L_0 = \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{4}\varepsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle.$$

Φ is a superfield $\Phi\colon \mathbb{R}^{2|2} \to M$, and $\langle\cdot,\cdot\rangle$ is the metric on M pulled back to a metric on Φ^*TM. Then, according to [7] the supersymmetric Lagrangian L, given in (15), is obtained by integrating over the θ variables the Lagrangian density:

$$L = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \frac{1}{4}\varepsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle.$$

Let us compute the variation of L_0 under an arbitrary even variation $\delta\Phi$ of the superfield Φ. We will set $\nabla_{D_a} = D_a^{\nabla}$. Then, following [7], we have

$$\begin{aligned}\delta L_0 &= \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{4}\varepsilon^{ab}(\langle\delta_\nabla D_a\Phi, D_b\Phi\rangle + \langle D_a\Phi, \delta_\nabla D_b\Phi\rangle)\\ &= \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{2}\varepsilon^{ab}\langle\delta_\nabla D_a\Phi, D_b\Phi\rangle\\ &= \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{2}\varepsilon^{ab}\langle D_a^\nabla\delta_\nabla\Phi, D_b\Phi\rangle\\ &= \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{2}\varepsilon^{ab}(D_a\langle\delta\Phi, D_b\Phi\rangle - \langle\delta\Phi, D_a^\nabla D_b\Phi\rangle)\\ &= \mathrm{d}\left[\iota(D_a)\left(\mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{2}\varepsilon^{ab}\langle D_b\Phi, \delta\Phi\rangle\right)\right] - \mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2\frac{1}{2}\langle\delta\Phi, (D_1^\nabla D_2 - D_2^\nabla D_1)\Phi\rangle.\end{aligned}$$

We have used in the last stage the fact that the density $\mathrm{d}x\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2$ is invariant under D_a and the Cartan formula for the Lie derivative. So the Euler–Lagrange equation in the superspace is

$$(D_1^\nabla D_2 - D_2^\nabla D_1)\Phi = 0$$

or equivalently,

$$\bar{D}^\nabla D\Phi = 0. \tag{19}$$

3. Lift of a superharmonic map into a symmetric space

3.1. The case $M = S^n$

We consider the quotient map $\pi\colon \mathrm{SO}(n+1) \to S^n$ defined by $\pi(v_1,\ldots,v_{n+1}) = v_{n+1}$. We will say that $\mathcal{F}\colon \mathbb{R}^{2|2} \to \mathrm{SO}(n+1)$ is a lift of $\Phi\colon \mathbb{R}^{2|2} \to S^n$ if $\pi\circ\mathcal{F} = \Phi$. Let

$$\mathcal{F} = U + \theta_1\Psi_1 + \theta_2\Psi_2 + \theta_1\theta_2 f$$

be the writing of $\mathcal{F}$ in $\mathfrak{M}_{n+1}(\mathbb{R})$ (the algebra of $(n+1)\times(n+1)$-matrices) and write that ${}^t\mathcal{F}\mathcal{F} = \mathbf{1}$ (this means that if $h := A \in \mathfrak{M}_{n+1}(\mathbb{R}) \mapsto {}^tAA - \mathbf{1} \in \mathfrak{M}_{n+1}(\mathbb{R})$, then $\mathcal{F}^*(h) = h\circ\mathcal{F} = 0$); we get

$$\begin{aligned}&{}^tU\,U = Id\\ &A_i = U^{-1}\Psi_i \text{ is antisymmetric: } {}^tA_i = -A_i\\ &{}^tU\,f + {}^tfU - {}^t\Psi_1\,\Psi_2 + {}^t\Psi_2\,\Psi_1 = 0.\end{aligned}$$

The third equation can be rewritten, setting $B = U^{-1}f$ and using ${}^tA_i = -A_i$, as

$$B + {}^tB + A_1A_2 - A_2A_1 = 0.$$

Now we consider the Maurer–Cartan form of $\mathcal{F}$:

$$\alpha = \mathcal{F}^{-1}\mathrm{d}\mathcal{F} = {}^t\mathcal{F}\,\mathrm{d}\mathcal{F}.$$

We can write

$$0 = \mathrm{d}({}^t\mathcal{F}\mathcal{F}) = (\mathrm{d}\,{}^t\mathcal{F})\mathcal{F} + {}^t\mathcal{F}\,\mathrm{d}\mathcal{F} = {}^t\alpha + \alpha,$$

so α is a 1-form on $\mathbb{R}^{2|2}$ with values in $\mathrm{so}(n+1)$.

Take the exterior derivative of $\mathrm{d}\mathcal{F} = \mathcal{F}\alpha$, we get

$$0 = \mathrm{d}(\mathrm{d}\mathcal{F}) = \mathrm{d}\mathcal{F} \wedge \alpha + \mathcal{F}\mathrm{d}\alpha = \mathcal{F}(\alpha \wedge \alpha + \mathrm{d}\alpha).$$

Hence since $\mathcal{F}$ is invertible (${}^t\mathcal{F}\mathcal{F} = \mathbf{1}$)

$$\mathrm{d}\alpha + \alpha \wedge \alpha = 0.$$

We write $\mathrm{so}(n+1) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the Cartan decomposition of $\mathrm{so}(n+1)$. We have $\mathfrak{g}_0 = \mathrm{so}(n)$ and $\mathfrak{g}_1 = \left\{ \begin{pmatrix} \mathrm{O} & v \\ -{}^t v & 0 \end{pmatrix}, v \in \mathbb{R}^n \right\}$. We will write $\alpha = \alpha_0 + \alpha_1$, the decomposition of α.

We want to write the Euler–Lagrange equation (18) in terms of α. Setting $X = \mathcal{F}^{-1} D\Phi$ then $\alpha_1(D) = \begin{pmatrix} \mathrm{O} & X \\ -{}^t X & 0 \end{pmatrix}$ and so we have

$$\begin{aligned} \bar{D}X = \bar{D}(\mathcal{F}^{-1}D\Phi) &= (\bar{D}\,{}^t\mathcal{F})\mathcal{F}X + \mathcal{F}^{-1}(\bar{D}D\Phi) \\ &= {}^t\alpha(D)X + \mathcal{F}^{-1}(\bar{D}D\Phi). \end{aligned}$$

That is,

$$\mathcal{F}^{-1}(\bar{D}D\Phi) = \bar{D}X + \alpha(\bar{D})X. \tag{20}$$

Moreover

$$\mathcal{F}^{-1}(\langle \bar{D}\Phi, D\Phi\rangle \Phi) = \langle \bar{D}\Phi, D\Phi\rangle e_{n+1} = \langle \bar{X}, X\rangle e_{n+1} \tag{21}$$

and the last equality results from the fact that $\mathcal{F}$ is a map into $\mathrm{SO}(n+1)$; $(e_i)_{1\le i\le n+1}$ is the canonical basis of $\mathbb{R}^{n+1}$. Besides we have

$$\alpha(\bar{D})X = \begin{pmatrix} \alpha_0(\bar{D}) & \bar{X} \\ -{}^t\bar{X} & 0 \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_0(\bar{D})X \\ -\langle \bar{X}, X\rangle \end{pmatrix}. \tag{22}$$

Hence, combining (20)–(22), we obtain that the Eq. (18) can be written in terms of α as

$$\bar{D}X + \alpha_0(\bar{D})X = 0,$$

or equivalently

$$\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)] = 0$$

where [,] is the supercommutator. Thus, we have the following:

Theorem 3. *Let $\Phi\colon \mathbb{R}^{2|2} \to S^n$ be a superfield with lift $\mathcal{F}\colon \mathbb{R}^{2|2} \to \mathrm{SO}(n+1)$; then Φ is superharmonic if and only if the Maurer–Cartan form $\alpha = \mathcal{F}^{-1}\mathrm{d}\mathcal{F} = \alpha_0 + \alpha_1$ satisfies*

$$\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)] = 0.$$

3.2. The general case

We suppose that $M = G/H$ is a Riemannian symmetric space with symmetric involution $\tau\colon G \to G$ so that $G^\tau \supset H \supset (G^\tau)_0$. Let $\pi\colon G \to M$ be the canonical projection and let $\mathfrak{g}$, $\mathfrak{g}_0$ be the Lie algebras of G and H respectively. Write $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the Cartan decomposition, with the commutator relations $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \bmod 2}$.

Recall that the tangent bundle TM is canonically isomorphic to the subbundle $[\mathfrak{g}_1]$ of the trivial bundle $M \times \mathfrak{g}$, with fibre $\mathrm{Ad}g(\mathfrak{g}_1)$ over the point $x = g\cdot H \in M$. Under this identification the Levi-Civita connection of M is just the flat differentiation in $M \times \mathfrak{g}$ followed by the projection on $[\mathfrak{g}_1]$ along $[\mathfrak{g}_0]$ (which is defined in the same way as $\mathfrak{g}_1$; see [4,8,2,18]). Let $\Phi\colon \mathbb{R}^{2|2} \to M$ be a superfield with lift $\mathcal{F}\colon \mathbb{R}^{2|2} \to G$ so that $\pi \circ \mathcal{F} = \Phi$. Consider the Maurer–Cartan form of $\mathcal{F}$: $\alpha = \mathcal{F}^{-1}\cdot \mathrm{d}\mathcal{F}$. It is the pullback by $\mathcal{F}$ of the Maurer–Cartan form of the group G. It is a 1-form on $\mathbb{R}^{2|2}$ with values in the Lie algebra $\mathfrak{g}$. We decompose it in the form $\alpha = \alpha_0 + \alpha_1$, following the Cartan decomposition. Then the canonical isomorphism of a bundle between TM and $[\mathfrak{g}_1]$ leads to a isomorphism between $\Phi^*(TM)$ and $\Phi^*[\mathfrak{g}_1]$ and the image of $D\Phi$ under this isomorphism is $\mathrm{Ad}\mathcal{F}(\alpha_1(D))$. Thus the Euler–Lagrange equation (19) is written as

$$[\bar{D}(\,\mathrm{Ad}\mathcal{F}(\alpha_1(D))\,)]_{\Phi^*[\mathfrak{g}_1]} = 0$$

where $[\cdot]_{\Phi^*[\mathfrak{g}_1]}$ is the projection on $[\mathfrak{g}_1]$ along $[\mathfrak{g}_0]$, pulled back by Φ to the projection on $\Phi^*[\mathfrak{g}_1]$ along $\Phi^*[\mathfrak{g}_0]$. Using the fact that

$$A\colon (g,\eta) \in G \times \mathfrak{g} \mapsto \mathrm{Ad}g(\eta)$$

satisfies

$$\mathrm{d}A = \mathrm{Ad}g(\mathrm{d}\eta + [g^{-1}\cdot \mathrm{d}g, \eta]),$$

where $g^{-1}\cdot \mathrm{d}g$ is the Maurer–Cartan form of G, this equation becomes

$$\begin{aligned} 0 &= \left[\mathrm{Ad}\mathcal{F}\left(\bar{D}\alpha_1(D) + [\alpha(\bar{D}), \alpha_1(D)]\right)\right]_{\Phi^*[\mathfrak{g}_1]} \\ &= \mathrm{Ad}\mathcal{F}[\bar{D}\alpha_1(D) + [\alpha(\bar{D}), \alpha_1(D)]]_1 \\ &= \mathrm{Ad}\mathcal{F}\left(\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)]\right). \end{aligned}$$

So we arrive at the same characterization as in the particular case $M = S^n$.

Theorem 4. *A superfield $\Phi\colon \mathbb{R}^{2|2} \to M$ with lift $\mathcal{F}\colon \mathbb{R}^{2|2} \to G$ is superharmonic if and only if the Maurer–Cartan form $\alpha = \mathcal{F}^{-1}\cdot \mathrm{d}\mathcal{F} = \alpha_0 + \alpha_1$ satisfies*

$$\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)] = 0.$$

4. The zero-curvature equation

Lemma 1. *Each 1-form α on $\mathbb{R}^{2|2}$ can be written in the form*

$$\alpha = \mathrm{d}\theta\,\alpha(D) + \mathrm{d}\bar{\theta}\,\alpha(\bar{D}) + (\mathrm{d}z + (\mathrm{d}\theta)\theta)\,\alpha\left(\frac{\partial}{\partial z}\right) + (\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta})\,\alpha\left(\frac{\partial}{\partial \bar{z}}\right).$$

Proof. The dual basis of $\left\{D, \bar{D}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right\}$ is $\left\{\mathrm{d}\theta, \mathrm{d}\bar{\theta}, \mathrm{d}z + (\mathrm{d}\theta)\theta, \mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta}\right\}$. ■

We consider now that α is a 1-form on $\mathbb{R}^{2|2}$ with values in the Lie algebra $\mathfrak{g}$; then using the writing given by the lemma, we have

$$\begin{aligned} \mathrm{d}\alpha + \frac{1}{2}[\alpha\wedge\alpha] = {} & -\mathrm{d}\theta\wedge\mathrm{d}\theta\left\{D\alpha(D) + \frac{1}{2}[\alpha(D), \alpha(D)] + \alpha\left(\frac{\partial}{\partial z}\right)\right\} \\ & -\mathrm{d}\bar{\theta}\wedge\mathrm{d}\bar{\theta}\left\{\bar{D}\alpha(\bar{D}) + \frac{1}{2}[\alpha(\bar{D}), \alpha(\bar{D})] + \alpha\left(\frac{\partial}{\partial \bar{z}}\right)\right\} \\ & -\mathrm{d}\theta\wedge\mathrm{d}\bar{\theta}\left\{\bar{D}\alpha(D) + D\alpha(\bar{D}) + [\alpha(\bar{D}), \alpha(D)]\right\} \\ & + (\mathrm{d}z + (\mathrm{d}\theta)\theta)\wedge(\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta})\left\{\partial_z\alpha\left(\frac{\partial}{\partial \bar{z}}\right) - \partial_{\bar{z}}\alpha\left(\frac{\partial}{\partial z}\right) + \left[\alpha\left(\frac{\partial}{\partial z}\right), \alpha\left(\frac{\partial}{\partial \bar{z}}\right)\right]\right\} \\ & + (\mathrm{d}\theta)\wedge(\mathrm{d}z + (\mathrm{d}\theta)\theta)\left\{D\alpha\left(\frac{\partial}{\partial z}\right) - \partial_z\alpha(D) + \left[\alpha(D), \alpha\left(\frac{\partial}{\partial z}\right)\right]\right\} \\ & + \text{ conjugate expression} \\ & + \mathrm{d}\theta\wedge(\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta})\left\{D\alpha\left(\frac{\partial}{\partial \bar{z}}\right) - \partial_{\bar{z}}\alpha(D) + \left[\alpha(D), \alpha\left(\frac{\partial}{\partial \bar{z}}\right)\right]\right\} \\ & + \text{ conjugate expression.} \end{aligned} \tag{23}$$

In the following, we will write the terms like $\frac{1}{2}[\alpha(D),\alpha(D)]$ in the form $\alpha(D)^2$. This is justified by the fact that if we embed $\mathfrak{g}$ in a matrix algebra or more intrinsically in its universal enveloping algebra, so that we can write $[a,b]=ab-ba$, then the supercommutator is given by

$$[a,b]=ab-(-1)^{p(a)p(b)}ba,$$

p being the parity, and thus $[a,a]=2a^2$ if a is odd.

The following theorem characterizes the 1-forms on $\mathbb{R}^{2|2}$ which are Maurer–Cartan forms.

Theorem 5. • *Let α be a 1-form on $\mathbb{R}^{2|2}$ with values in the Lie algebra $\mathfrak{g}$ of the Lie group G. Then there exists $\mathcal{F}\colon\mathbb{R}^{2|2}\to G$ such that $\mathrm{d}\mathcal{F}=\mathcal{F}\alpha$ if and only if*

$$\mathrm{d}\alpha+\frac{1}{2}[\alpha\wedge\alpha]=0.$$

Moreover, if $U(z_0)$ is given then $\mathcal{F}$ is unique ($z_0\in\mathbb{R}^2$, $U=i^*\mathcal{F}$).

• *Let $A_D,\ A_{\bar D}\colon\mathbb{R}^{2|2}\to\mathfrak{g}\otimes\mathbb{C}$ be odd maps; then the two following statements are equivalent:*

$$\text{(i)}\ \ \exists\mathcal{F}\colon\mathbb{R}^{2|2}\to G^{\mathbb{C}}\qquad D\mathcal{F}=\mathcal{F}A_D,\ \bar D\mathcal{F}=\mathcal{F}A_{\bar D}\tag{24}$$

$$\text{(ii)}\ \ \bar DA_D+DA_{\bar D}+[A_{\bar D},A_D]=0.\tag{25}$$

Moreover $\mathcal{F}$ is unique if we are given $U(z_0)$, and $\mathcal{F}$ is with values in G if and only if $A_{\bar D}=\overline{A_D}$. In particular, the natural map

$$I_{(D,\bar D)}\colon\{\alpha\ 1\text{-}form\mid \mathrm{d}\alpha+\alpha\wedge\alpha=0\}\longrightarrow\{(A_D,A_{\bar D})\ odd\ which\ satisfy\ \text{(ii)}\}$$
$$\alpha\longmapsto(\alpha(D),\alpha(\bar D))$$

is a bijection.

Remark 3. • Suppose that $A_{\bar D}=\overline{A_D}$. If we embed $\mathfrak{g}$ in a matrix algebra then (ii) means that

$$\bar DA_D+DA_{\bar D}+A_{\bar D}A_D+A_DA_{\bar D}=0.$$

That is,

$$\mathrm{Re}(\bar DA_D+A_{\bar D}A_D)=0.$$

• We can see according to (23) that if $\mathrm{d}\alpha+\frac{1}{2}[\alpha\wedge\alpha]=0$ then $\alpha(\frac{\partial}{\partial z})$ (resp. $\alpha(\frac{\partial}{\partial\bar z})$) can be expressed in terms of $\alpha(D)$ (resp. $\alpha(\bar D)$):

$$\alpha\left(\frac{\partial}{\partial z}\right)=-(D\alpha(D)+\alpha(D)^2).\tag{26}$$

Proof of the Theorem 5. The first point follows from the Frobenius theorem (which holds in supermanifolds; see [6,20,21]), for the existence. For the uniqueness, if $\mathcal{F}$ and $\mathcal{F}'$ are solutions then $\mathrm{d}(\mathcal{F}'\mathcal{F}^{-1})=0$ so $\mathcal{F}'\mathcal{F}^{-1}$ is a constant $C\in G$, and $C=U'(z_0)U^{-1}(z_0)$.

For the second point, the implication (i) $\Longrightarrow$ (ii) follows from (23) (see the term in $\mathrm{d}\theta\wedge\mathrm{d}\bar\theta$). Let us prove (ii) $\Longrightarrow$ (i).

A_D and $A_{\bar D}$ are odd maps from $\mathbb{R}^{2|2}$ into $\mathfrak{g}\otimes\mathbb{C}$ so let us write

$$A_D=A_D^0+\theta A_D^\theta+\bar\theta A_D^{\bar\theta}+\theta\bar\theta A_D^{\theta\bar\theta}$$
$$A_{\bar D}=A_{\bar D}^0+\theta A_{\bar D}^\theta+\bar\theta A_{\bar D}^{\bar\theta}+\theta\bar\theta A_{\bar D}^{\theta\bar\theta}$$

so that we have

$$\bar DA_D=A_D^{\bar\theta}-\theta A_D^{\theta\bar\theta}-\bar\theta\frac{\partial A_D^0}{\partial\bar z}+\theta\bar\theta\frac{\partial A_D^\theta}{\partial\bar z}$$

$$DA_{\bar D}=A_{\bar D}^\theta+\bar\theta A_{\bar D}^{\theta\bar\theta}-\theta\frac{\partial A_{\bar D}^0}{\partial z}-\theta\bar\theta\frac{\partial A_{\bar D}^{\bar\theta}}{\partial z}.$$

Thus the Eq. (25) splits into four equations:

$$\begin{aligned}
&A_D^{\bar\theta} + A_{\bar D}^{\theta} + [A_{\bar D}^0, A_D^0] = 0\\
&-A_D^{\theta\bar\theta} - \frac{\partial A_{\bar D}^0}{\partial z} + [A_{\bar D}^{\theta}, A_D^0] + [A_D^{\theta}, A_{\bar D}^0] = 0\\
&A_{\bar D}^{\theta\bar\theta} - \frac{\partial A_D^0}{\partial \bar z} + [A_D^{\bar\theta}, A_{\bar D}^0] + [A_{\bar D}^{\bar\theta}, A_D^0] = 0\\
&\frac{\partial A_D^{\theta}}{\partial \bar z} - \frac{\partial A_{\bar D}^{\bar\theta}}{\partial z} + [A_D^0, A_{\bar D}^{\theta\bar\theta}] + [A_{\bar D}^0, A_D^{\theta\bar\theta}] + [A_D^{\theta}, A_{\bar D}^{\bar\theta}] + [A_{\bar D}^{\theta}, A_D^{\bar\theta}] = 0.
\end{aligned} \tag{27}$$

Now, let us embed $\mathfrak{g}$ in a matrix algebra $\mathfrak{M}_m(\mathbb{R})$; then the Lie bracket in $\mathfrak{g}$ is given by $[a,b] = ab - ba$. Let us define $A, \underline{A}, \beta, B, \underline{B}$ by

$$\begin{aligned}
&A = A_D^0, \qquad \underline{A} = A_{\bar D}^0, \qquad A_D^{\theta} = -\beta\left(\frac{\partial}{\partial z}\right) - A^2, \qquad A_{\bar D}^{\bar\theta} = -\beta\left(\frac{\partial}{\partial \bar z}\right) - \underline{A}^2,\\
&\quad A_D^{\bar\theta} = B - \underline{A}A, \qquad A_{\bar D}^{\theta} = \underline{B} - A\underline{A},
\end{aligned} \tag{28}$$

and then the four previous Eq. (27) are written as

$$B + \underline{B} = 0 \tag{29}$$

$$A_D^{\theta\bar\theta} = -\frac{\partial \underline{A}}{\partial z} + [-B - A\underline{A}, A] + \left[-\beta\left(\frac{\partial}{\partial z}\right) - A^2, \underline{A}\right] \tag{30}$$

$$A_{\bar D}^{\theta\bar\theta} = \frac{\partial A}{\partial \bar z} + [\underline{A}, B - \underline{A}A] + \left[A, -\beta\left(\frac{\partial}{\partial \bar z}\right) - \underline{A}^2\right] \tag{31}$$

$$\begin{aligned}
&\frac{\partial}{\partial z}\beta\left(\frac{\partial}{\partial \bar z}\right) - \frac{\partial}{\partial \bar z}\beta\left(\frac{\partial}{\partial z}\right) + \frac{\partial \underline{A}^2}{\partial z} - \frac{\partial A^2}{\partial \bar z} + \left[A, \frac{\partial A}{\partial \bar z} + [\underline{A}, B - \underline{A}A] + \left[A, -\beta\left(\frac{\partial}{\partial \bar z}\right) - \underline{A}^2\right]\right]\\
&\quad + \left[\underline{A}, -\frac{\partial \underline{A}}{\partial z} + [-B - A\underline{A}, A] + \left[-\beta\left(\frac{\partial}{\partial z}\right) - A^2, \underline{A}\right]\right]\\
&\quad + \left[-\beta\left(\frac{\partial}{\partial \bar z}\right) - \underline{A}^2, -\beta\left(\frac{\partial}{\partial \bar z}\right) - \underline{A}^2\right] + \left[-B - A\underline{A}, B - \underline{A}A\right] = 0.
\end{aligned} \tag{32}$$

The last equation becomes after simplification

$$\frac{\partial}{\partial z}\beta\left(\frac{\partial}{\partial \bar z}\right) - \frac{\partial}{\partial \bar z}\beta\left(\frac{\partial}{\partial z}\right) + \left[\beta\left(\frac{\partial}{\partial z}\right), \beta\left(\frac{\partial}{\partial \bar z}\right)\right] = 0$$

so since β is even and with values in $\mathfrak{g}^{\mathbb{C}}$ (resp. in $\mathfrak{g}$ if $A_{\bar D} = \overline{A_D}$), according to (28), we deduce from this that there exists $U : \mathbb{R}^{2|2} \to G^{\mathbb{C}}$ such that $U^{-1}\mathrm{d}U = \beta$ and U is unique if $U(z_0)$ is given, and with values in G if $A_{\bar D} = \overline{A_D}$. Then we set[1]

$$\frac{1}{2}\Psi = UA, \qquad \frac{1}{2}\underline{\Psi} = U\underline{A}, \qquad f = \frac{2}{i}UB \tag{33}$$

and

$$\mathcal{F} = U + \frac{1}{2}(\theta\Psi + \bar\theta\underline{\Psi}) + \frac{i}{2}\theta\bar\theta f. \tag{34}$$

The result $\mathcal{F}$ is a superfield from $\mathbb{R}^{2|2}$ into $\mathfrak{M}_m(\mathbb{C})$ and according to (6) (with $\mathbb{R}^N = \mathfrak{M}_m(\mathbb{C})$, $M = \mathrm{GL}_m(\mathbb{C})$, $f_\alpha = 0$, $U_\alpha = M$) since U is invertible and hence with values in $\mathrm{GL}_m(\mathbb{C})$, $\mathcal{F}$ takes values in $\mathrm{GL}_m(\mathbb{C})$. Besides, it takes values

[1] See Remark 4.

in $\mathrm{GL}_m(\mathbb{R})$ if $A_{\bar{D}} = \overline{A_D}$. We compute that

$$\begin{aligned}
\mathcal{F}^{-1} &= \left(U + \left[\frac{1}{2}(\theta\Psi + \bar{\theta}\underline{\Psi}) + \frac{i}{2}\theta\bar{\theta}f\right]\right)^{-1} \\
&= \sum_{k=0}^{2}(-1)^k \left[U^{-1}\left(\frac{1}{2}(\theta\Psi + \bar{\theta}\underline{\Psi}) + \frac{i}{2}\theta\bar{\theta}f\right)\right]^k U^{-1} \\
&= \left[\mathbf{1} - (\theta A + \bar{\theta}\underline{A}) - \theta\bar{\theta}B + \theta A\bar{\theta}\underline{A} + \bar{\theta}\underline{A}\theta A\right] U^{-1} \\
&= \left[\mathbf{1} - \theta A - \bar{\theta}\underline{A} - \theta\bar{\theta}(B + A\underline{A} - \underline{A}A)\right] U^{-1}
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{F}^{-1} \cdot D\mathcal{F} &= \mathcal{F}^{-1}\left(\frac{1}{2}\Psi - \theta\frac{\partial U}{\partial z} + \frac{i}{2}\bar{\theta}f - \theta\bar{\theta}\frac{\partial\underline{\Psi}}{\partial z}\right) \\
&= A + \theta\left(-\beta\left(\frac{\partial}{\partial z}\right) - A^2\right) + \bar{\theta}(B - \underline{A}A) \\
&\quad + \theta\bar{\theta}\left(-\frac{\partial\underline{A}}{\partial z} - \beta\left(\frac{\partial}{\partial z}\right)\underline{A} - (B + A\underline{A} - \underline{A}A)A + AB + \underline{A}\beta\left(\frac{\partial}{\partial z}\right)\right) \\
&= A + \theta\left(-\beta\left(\frac{\partial}{\partial z}\right) - A^2\right) + \bar{\theta}(B - \underline{A}A) \\
&\quad + \theta\bar{\theta}\left(-\frac{\partial\underline{A}}{\partial z} + [-B - A\underline{A}, A] + \left[-\beta\left(\frac{\partial}{\partial z}\right) - A^2, \underline{A}\right]\right)
\end{aligned}$$

and thus according to (28) and (30) we conclude that

$$\mathcal{F}^{-1} \cdot D\mathcal{F} = A_D.$$

We can check in the same way that $\mathcal{F}^{-1} \cdot \bar{D}\mathcal{F} = A_{\bar{D}}$. Moreover if we consider $\alpha = \mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}$, the Maurer–Cartan form of $\mathcal{F}$, then $(\alpha(D), \alpha(\bar{D})) = (A_D, A_{\bar{D}})$ is with values in $\mathfrak{g}^{\mathbb{C}}$, and hence it holds also for $\alpha\left(\frac{\partial}{\partial z}\right), \alpha\left(\frac{\partial}{\partial \bar{z}}\right)$ according to (26). So α takes values in $\mathfrak{g}^{\mathbb{C}}$. But, according to the first point of the theorem, the equation $\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F} = \alpha$ has a unique solution if $U(z_0)$ is given, and this solution is with values in $G^{\mathbb{C}}$ since α takes values in $\mathfrak{g}^{\mathbb{C}}$ and $U(z_0)$ is in $G^{\mathbb{C}}$. So $\mathcal{F}$ takes values in $G^{\mathbb{C}}$. Moreover, $\mathcal{F}$ takes values in G if $A_{\bar{D}} = \overline{A_D}$. Hence, the map $I_{(D,\bar{D})}$ is surjective. Besides it is injective by the second point of the Remark 3: according to (26), α is completely determined by $(\alpha(D), \alpha(\bar{D}))$. We have proved the theorem. ■

Remark 4. In general, G is not embedded in $\mathrm{GL}_m(\mathbb{R})$. But since $\mathfrak{g}$ is embedded in $\mathfrak{M}_m(\mathbb{R})$, there exists a unique morphism of group, which is an immersion, $j: G \to \mathrm{GL}_m(\mathbb{R})$, the image of which is the subgroup generated by $\exp(\mathfrak{g})$. In other words G is an integral subgroup of $\mathrm{GL}_m(\mathbb{R})$ (and not a closed subgroup). In the demonstration we use the abuse of language consisting in identifying G and $j(G)$. For example in (33) and (34) we must use $j \circ U$ instead of U; and at the end of the demonstration, when we use the first point of the theorem, we must say that there exists a unique solution with values in G, $\mathcal{F}_1$, and by the uniqueness of the solution (in $\mathrm{GL}_m(\mathbb{R})$) we have $j \circ \mathcal{F}_1 = \mathcal{F}$. However, in the case which interests us, G is semi-simple so it can be represented as a subgroup of $\mathrm{GL}_m(\mathbb{R})$ via the adjoint representation, and so there is no ambiguity in this case.

Remark 5. To our knowledge, this theorem (more precisely the implication (ii) $\Longrightarrow$ (i)) has never be demonstrated in the literature. We have only found a statement without any proof, of this, in [22].

Now we are able to prove:

Theorem 6. *Let $\Phi: \mathbb{R}^{2|2} \to M = G/H$ be a superfield into a symmetric space with lift $\mathcal{F}: \mathbb{R}^{2|2} \to G$ and Maurer–Cartan form $\alpha = \mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}$; then the following statements are equivalent:*

(i) *Φ is superharmonic.*

(ii) *Setting* $\alpha(D)_\lambda = \alpha_0(D) + \lambda^{-1}\alpha_1(D)$ *and* $\alpha(\bar{D})_\lambda = \overline{\alpha(D)_\lambda} = \alpha_0(\bar{D}) + \lambda\alpha_1(\bar{D})$, *we have*

$$\bar{D}\alpha(D)_\lambda + D\alpha(\bar{D})_\lambda + [\alpha(\bar{D})_\lambda, \alpha(D)_\lambda] = 0, \quad \forall\lambda \in S^1.$$

(iii) *There exists a lift* $\mathcal{F}_\lambda : \mathbb{R}^{2|2} \to G$ *such that* $\mathcal{F}_\lambda^{-1} \cdot D\mathcal{F}_\lambda = \alpha_0(D) + \lambda^{-1}\alpha_1(D)$, *for all* $\lambda \in S^1$.

Then, in this case, for all $\lambda \in S^1$, $\Phi_\lambda = \pi \circ \mathcal{F}_\lambda$ *is superharmonic.*

Proof. Let us split Eq. (25) into the sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$:

$$\begin{cases} \bar{D}\alpha_0(D) + D\alpha_0(\bar{D}) + [\alpha_0(\bar{D}), \alpha_0(D)] + [\alpha_1(\bar{D}), \alpha_1(D)] = 0 \\ \mathrm{Re}(\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)]) = 0 \end{cases}$$

and so (ii) means that

$$\forall\lambda \in S^1, \quad \mathrm{Re}(\lambda^{-1}(\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)])) = 0$$

which means that

$$\bar{D}\alpha_1(D) + [\alpha_0(\bar{D}), \alpha_1(D)] = 0$$

and hence (i) $\iff$ (ii), according to Theorem 4. Moreover according to Theorem 5, (ii) and (iii) are equivalent. That completes the proof. ∎

We know that the extended Maurer–Cartan form, α_λ, given by the previous theorem is defined by $\alpha_\lambda(D) = \alpha_0(D) + \lambda^{-1}\alpha_1(D)$ and (so) $\alpha_\lambda(\bar{D}) = \alpha_0(\bar{D}) + \lambda\alpha_1(\bar{D})$. However we want to know how the other coefficients of α are transformed into coefficients of α_λ. From (26) we deduce

$$D\alpha_0(D) + \alpha_0(D)^2 + \alpha_1(D)^2 = -\alpha_0\left(\frac{\partial}{\partial z}\right)$$
$$D\alpha_1(D) + [\alpha_0(D), \alpha_1(D)] = -\alpha_1\left(\frac{\partial}{\partial z}\right)$$

and hence

$$(\alpha_\lambda)_0\left(\frac{\partial}{\partial z}\right) = \alpha_0\left(\frac{\partial}{\partial z}\right) + (1-\lambda^{-2})\,\alpha_1(D)^2$$
$$(\alpha_\lambda)_1\left(\frac{\partial}{\partial z}\right) = \lambda^{-1}\alpha_1\left(\frac{\partial}{\partial z}\right).$$

Finally we have

$$\begin{aligned}\alpha_\lambda = {} & -\lambda^{-2}\alpha_1(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta) + \lambda^{-1}\alpha_1' + \alpha_0 + 2\mathrm{Re}\left(\alpha_1(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta)\right) + \lambda\alpha_1'' \\ & - \lambda^2\alpha_1(\bar{D})^2(\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta})\end{aligned} \tag{35}$$

where

$$\begin{aligned}\alpha_1' &= \mathrm{d}\theta\alpha_1(D) + (\mathrm{d}z + (\mathrm{d}\theta)\theta)\,\alpha_1\left(\frac{\partial}{\partial z}\right) \\ \alpha_1'' &= \mathrm{d}\bar{\theta}\alpha_1(\bar{D}) + (\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta})\,\alpha_1\left(\frac{\partial}{\partial \bar{z}}\right).\end{aligned} \tag{36}$$

So, we remark that, contrary to the classical case for harmonic maps $u : \mathbb{R}^2 \to G/H$, where the extended Maurer–Cartan form is given by $\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$ (see [8]), here in the supersymmetric case we obtain terms on λ^{-2} and λ^2, and the term on λ^0 is $\alpha_0 + 2\mathrm{Re}\,(\alpha(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta))$ instead of α_0. Moreover, since $\alpha_1(D)^2 = \frac{1}{2}[\alpha_1(D), \alpha_1(D)]$ takes values in $\mathfrak{g}_0^{\mathbb{C}}$, we conclude that $(\alpha_\lambda)_{\lambda\in S^1}$ is a 1-form on $\mathbb{R}^{2|2}$ with values in

$$\Lambda\mathfrak{g}_\tau = \{\xi : S^1 \to \mathfrak{g} \text{ smooth} \mid \xi(-\lambda) = \tau(\xi(\lambda))\}$$

(see [8] or [23] for more details for loop groups and their Lie algebras). And so the extended lift $(\mathcal{F}_\lambda)_{\lambda\in S^1}\colon \mathbb{R}^{2|2} \to \Lambda G$ leads to a map $(\mathcal{F}_\lambda)_{\lambda\in S^1}\colon \mathbb{R}^{2|2} \to \Lambda G_\tau$. As in [8], for the classical case, this yields the following characterization of superharmonic maps $\Phi\colon \mathbb{R}^{2|2} \to G/H$.

Corollary 1. *A map $\Phi\colon \mathbb{R}^{2|2} \to G/H$ is superharmonic if and only if there exists a map $(\mathcal{F}_\lambda)_{\lambda\in S^1}\colon \mathbb{R}^{2|2} \to \Lambda G_\tau$ such that $\pi\circ\mathcal{F}_1 = \Phi$ and*

$$\mathcal{F}_\lambda^{-1}\cdot \mathrm{d}\mathcal{F}_\lambda = -\lambda^{-2}\alpha_1(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta) + \lambda^{-1}\alpha_1' + \tilde{\alpha}_0 + \lambda\alpha_1'' - \lambda^2\alpha_1(\bar{D})^2(\mathrm{d}\bar{z} + (\mathrm{d}\bar{\theta})\bar{\theta}),$$

where $\tilde{\alpha}_0$ and α_1 are $\mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{g}_1^{\mathbb{C}}$-valued, respectively, 1-forms on $\mathbb{R}^{2|2}$, and α_1', α_1'' are given by (36). Such a $(\mathcal{F}_\lambda)$ will be called an extended (superharmonic) lift.

Remark 6. Our result for the Maurer–Cartan form (35) is different from the one obtained in [15,17] or in [19]. Because in these papers, we have a decomposition $\mathfrak{g} = \bigoplus_{i=0}^3 \mathfrak{g}_i$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\hat{\alpha}_2$, the coefficient on λ^2, is independent of $\hat{\alpha}_1$ whereas here we have $\hat{\alpha}_2 = -\hat{\alpha}_1(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta)$. As we can see, in Theorem 6, if we decide to identify all the Maurer–Cartan forms with their images by $I_{(D,\bar{D})}$, $(\alpha(D), \alpha(\bar{D}))$, then the terms on λ^2 and λ^{-2} disappear and the things are analogous to the classical case. In other words, it is possible to have the same formulation of the results as for the classical case if we choose to work on $(\alpha(D), \alpha(\bar{D}))$ instead of working on the Maurer–Cartan form α. But as we will see, in the Weierstrass representation one cannot get rid completely of the terms on λ^2 and λ^{-2}. So these terms are not anecdotal and constitute an essential difference between the supersymmetric case and the classical one.

Remark 7. In the following, we will simply denote by $\mathcal{F}$ the extended lift $(\mathcal{F}_\lambda)\colon \mathbb{R}^{2|2} \to \Lambda G_\tau$; there is no ambiguity because we will always make precise where $\mathcal{F}$ takes values by writing $\mathcal{F}\colon \mathbb{R}^{2|2} \to \Lambda G_\tau$. Besides, given a superharmonic map $\Phi\colon \mathbb{R}^{2|2} \to G/H$, an extended lift $\mathcal{F}\colon \mathbb{R}^{2|2} \to \Lambda G_\tau$ is determined only up to a gauge transformation $K\colon \mathbb{R}^{2|2} \to H$ because $\mathcal{F}H$ is also an extended lift for Φ. Then following [8], we denote by $\mathcal{SH}$ the set

$$\mathcal{SH} = \{\Phi\colon \mathbb{R}^{2|2} \to G/H \text{ superharmonic}, i^*\Phi(0) = \pi(1)\}$$

and then we have a bijective correspondence between $\mathcal{SH}$ and

$$\{\mathcal{F}\colon \mathbb{R}^{2|2} \to \Lambda G_\tau,\ \text{extended lift}, i^*\mathcal{F}(0) \in H\}/C^\infty(\mathbb{R}^{2|2}, H).$$

We will define $\Phi = [\mathcal{F}]$.

5. Weierstrass-type representation of superharmonic maps

In this section, we shall show how we can use the method of [8] to obtain every superharmonic map $\Phi\colon \mathbb{R}^{2|2} \to G/H$ from Weierstrass-type data.

We recall the following (see [8,23]):

Theorem 7. *Assume that G is a compact semi-simple Lie group, $\tau\colon G \to G$ an order k automorphism of G with fixed point subgroup $G^\tau = H$. Let $H^{\mathbb{C}} = H\cdot\mathcal{B}$ be an Iwasawa decomposition for $H^{\mathbb{C}}$. Then:*

(i) *Multiplication $\Lambda G_\tau \times \Lambda^+_{\mathcal{B}} G^{\mathbb{C}}_\tau \xrightarrow{\sim} \Lambda G^{\mathbb{C}}_\tau$ is a diffeomorphism, onto.*

(ii) *Multiplication $\Lambda^-_* G^{\mathbb{C}}_\tau \times \Lambda^+ G^{\mathbb{C}}_\tau \longrightarrow \Lambda G^{\mathbb{C}}_\tau$ is a diffeomorphism onto the open and dense set $\mathcal{C} = \Lambda^-_* G^{\mathbb{C}}_\tau \cdot \Lambda^+ G^{\mathbb{C}}_\tau$, called the big cell.*

The above loop groups are defined by

$$\Lambda^+ G^{\mathbb{C}}_\tau = \{[\lambda \mapsto U_\lambda] \in \Lambda G^{\mathbb{C}}_\tau \ \text{ extending holomorphically in the unit disc}\}$$
$$\Lambda^+_{\mathcal{B}} G^{\mathbb{C}}_\tau = \{[\lambda \mapsto U_\lambda] \in \Lambda^+ G^{\mathbb{C}}_\tau \mid U(0) \in \mathcal{B}\}$$
$$\Lambda^-_* G^{\mathbb{C}}_\tau = \{[\lambda \mapsto U_\lambda] \in \Lambda G^{\mathbb{C}}_\tau \ \text{ extending holomorphically in the complement of the unit disc and } U_\infty = 0\}.$$

In analogous way one defines the corresponding Lie algebras $\Lambda\mathfrak{g}_\tau$, $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$, $\Lambda_*^-\mathfrak{g}_\tau^{\mathbb{C}}$ and $\Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}}$ where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B}$. Further we introduce

$$\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}} := \left\{ \xi \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}} \mid \xi_\lambda = \sum_{k=-2}^{+\infty} \lambda^k \xi_k \right\}.$$

Definition 2. We will say that a map $f \colon \mathbb{R}^{2|2} \to M$ is holomorphic if $\bar{D}f = 0$. We will say also that a 1-form μ on $\mathbb{R}^{2|2}$ is holomorphic if $\mu(\bar{D}) = 0$ and $\bar{D}\mu(D) = 0$. Moreover we will say that μ is a holomorphic potential if μ is a holomorphic 1-form on $\mathbb{R}^{2|2}$ with values in the Banach space $\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$ and if, writing $\mu = \sum_{k\geq -2} \lambda^k \mu_k$, we have $\mu_{-2}(D) = 0$. Then noticing that a holomorphic 1-form satisfies (25), we can say that the vector space $\mathcal{SP}$ of holomorphic potentials is

$$\mathcal{SP} = I_{(D,\bar{D})}{}^{-1}\{(\mu(D), 0) \mid \mu(D) \colon \mathbb{R}^{2|2} \to \Lambda_{-1,\infty}\mathfrak{g}_\tau^{\mathbb{C}} \text{ is odd, and } \bar{D}\mu(D) = 0\}.$$

Besides for a Maurer–Cartan form μ on $\mathbb{R}^{2|2}$ (in particular for a holomorphic 1-form) with values in $\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$ the condition $\mu_{-2}(D) = 0$ is equivalent to $\mu_{-2}\left(\frac{\partial}{\partial z}\right) = -(\mu_{-1}(D))^2$ according to (26).

As for the classical case (see [8]), we can construct superharmonic maps from a holomorphic potential: if $\mu \in \mathcal{SP}$ then μ satisfies (25), so we can integrate it:

$$g_\mu^{-1} \cdot \mathrm{d}g_\mu = \mu, \quad i^*g(0) = 1$$

to obtain a map $g_\mu \colon \mathbb{R}^{2|2} \to \Lambda G_\tau^{\mathbb{C}}$. We can decompose g_μ according to Theorem 7:

$$g_\mu = \mathcal{F}_\mu h_\mu$$

to obtain a map $\mathcal{F}_\mu \colon \mathbb{R}^{2|2} \to \Lambda G_\tau$ with $i^*\mathcal{F}_\mu(0) = 1$.

Theorem 8. *$\mathcal{F}_\mu \colon \mathbb{R}^{2|2} \to \Lambda G_\tau$ is an extended superharmonic lift.*

Proof. We have (dropping the index μ)

$$\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F} = \mathrm{Ad}h(\mu) - \mathrm{d}h \cdot h^{-1}.$$

But h takes values in $\Lambda_{\mathcal{B}}^+G_\tau^{\mathbb{C}}$ so that $\mathrm{d}h \cdot h^{-1}$ takes values in $\Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}}$, and hence

$$\left[\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}\right]_{\Lambda_*^-\mathfrak{g}_\tau^{\mathbb{C}}} = [\mathrm{Ad}h(\mu)]_{\Lambda_*^-\mathfrak{g}_\tau^{\mathbb{C}}}$$

is in the form

$$-\lambda^{-2}\alpha_1'(D)^2(\mathrm{d}z + (\mathrm{d}\theta)\theta) + \lambda^{-1}\alpha_1'$$

by using the Definition 2 of a holomorphic potential. But according to the reality condition contained in the definition of $\Lambda\mathfrak{g}_\tau$:

$$\left[\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}\right]_{\Lambda_*^+\mathfrak{g}_\tau^{\mathbb{C}}} = \overline{\left[\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}\right]_{\Lambda_*^-\mathfrak{g}_\tau^{\mathbb{C}}}},$$

we conclude that $\mathcal{F}^{-1} \cdot \mathrm{d}\mathcal{F}$ is in the same form as in the Corollary 1, so $\mathcal{F}$ is an extended superharmonic lift. ∎

Then according to the previous theorem we have defined a map

$$\mathcal{SW} \colon \mathcal{SP} \to \mathcal{SH} : \mu \mapsto [\mathcal{F}_\mu].$$

Theorem 9. *The map $\mathcal{SW} \colon \mathcal{SP} \to \mathcal{SH}$ is surjective and its fibres are the orbits of the based holomorphic gauge group*

$$\mathcal{G} = \{h \colon \mathbb{R}^{2|2} \to \Lambda^+G_\tau^{\mathbb{C}}, \bar{D}h = 0, i^*h(0) = 1\}$$

acting on $\mathcal{SP}$ by gauge transformations:

$$h \cdot \mu = \mathrm{Ad}h(\mu) - \mathrm{d}h \cdot h^{-1}.$$

Proof. As in [8] it is question of solving a $\bar{D}$-problem with right hand side in the Banach Lie algebra $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$:

$$\bar{D}h = -(\alpha_0(\bar{D}) + \lambda\alpha_1(\bar{D})) \cdot h \tag{37}$$

with $i^*h(0) = 1$. Let us embed $G^{\mathbb{C}}$ in $\mathrm{GL}_m(\mathbb{C})$ (G is semi-simple). Then we set

$$h = h_0 + \theta h_\theta + \bar{\theta} h_{\bar{\theta}} + \theta\bar{\theta} h_{\theta\bar{\theta}}$$

and $C = -(\alpha_0(\bar{D}) + \lambda\alpha_1(\bar{D})) = C_0 + \theta C_\theta + \bar{\theta} C_{\bar{\theta}} + \theta\bar{\theta} C_{\theta\bar{\theta}}$. These are respectively written in $\Lambda^+\mathfrak{M}_m(\mathbb{C})$ and in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$. Then (37) splits into

$$\begin{aligned}
&h_{\bar{\theta}} = C_0 h_0\\
&-h_{\theta\bar{\theta}} = -C_0 h_\theta + C_\theta h_0\\
&-\frac{\partial h_0}{\partial \bar{z}} = C_{\bar{\theta}} h_0 - C_0 h_{\bar{\theta}}\\
&\frac{\partial h_\theta}{\partial \bar{z}} = C_0 h_{\theta\bar{\theta}} + C_{\theta\bar{\theta}} h_0 + C_\theta h_{\bar{\theta}} - C_{\bar{\theta}} h_\theta
\end{aligned}$$

and hence we have for h_0

$$\frac{\partial h_0}{\partial \bar{z}} = -(C_{\bar{\theta}} - C_0^2) h_0.$$

This is a $\bar{\partial}$-problem with right hand side, $C_0^2 - C_\theta$, in the Banach Lie algebra $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$ which can be solved (see [8]). The solutions such that $h_0(0) = 1$ are determined only up to right multiplication by elements of

$$\mathcal{G}_0 = \{h_0 \colon \mathbb{R}^{2|2} \to \Lambda^+ G_\tau^{\mathbb{C}}, h_0(0) = 1,\ \partial_{\bar{z}} h_0 = 0\}.$$

Then $h_{\bar{\theta}}$ is given by $h_{\bar{\theta}} = C_0 h_0$ so it is tangent to $\Lambda^+ G_\tau^{\mathbb{C}}$ at h_0. $h_{\theta\bar{\theta}}$ is determined by h_0 and h_θ. So it remains to solve the equation on h_θ which can be rewritten, by expressing $h_{\theta\bar{\theta}}$ and $h_{\bar{\theta}}$ in terms of h_0 and h_θ at a first time, and by setting $h'_\theta = h_0^{-1} h_\theta$ at a second time, in the following way:

$$\frac{\partial h'_\theta}{\partial \bar{z}} = \left(\beta\left(\frac{\partial}{\partial \bar{z}}\right) + \mathrm{Ad}h_0^{-1}(C_0^2 - C_{\bar{\theta}})\right) h'_\theta + \mathrm{Ad}h_0^{-1}\left(C_{\theta\bar{\theta}} + [C_\theta, C_0]\right)$$

where $\beta = h_0^{-1}\mathrm{d}h_0$. Thus we obtain an equation of the form

$$\frac{\partial h'_\theta}{\partial \bar{z}} = a h'_\theta + b$$

with $a, b \colon \mathbb{R}^2 \to \Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$, which can be solved. The solutions such that $h'_\theta(0) = 0$ form an affine space whose the underlying vector space is

$$\left\{h'_\theta \colon \mathbb{R}^2 \to \Lambda^+\mathfrak{g}_\tau^{\mathbb{C}} \mid \frac{\partial h'_\theta}{\partial \bar{z}} = a h'_\theta,\ h'_\theta(0) = 0\right\}.$$

So we have solved (37). It remains to check that h is with values in $\Lambda^+ G_\tau^{\mathbb{C}}$. We know that h_0 takes values in $\Lambda^+ G_\tau^{\mathbb{C}}$; h_θ, $h_{\bar{\theta}}$ are tangent to $\Lambda^+ G_\tau^{\mathbb{C}}$ at h_0. It only remains for us to check that $h_{\theta\bar{\theta}}$ satisfies Eq. (7) (or (6)). But to do this we need to know more about the embedding $G^{\mathbb{C}} \hookrightarrow \mathrm{Gl}_m(\mathbb{C})$. It is possible to proceed like that (see Section 6), but we will follow another method.

Let $\gamma = \mathrm{d}h \cdot h^{-1}$ be the right Maurer–Cartan form of h. Then by (37), we have $\gamma(\bar{D}) = C$, and C takes values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$, so we have to prove that $\gamma(D)$ also takes values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$, in order to conclude that γ takes values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$ and finally that h takes values in $\Lambda^+ G_\tau^{\mathbb{C}}$, according to the first point of Theorem 5. Now return to the demonstration of the Theorem 5, where we put $\gamma(D) := A_D$, $\gamma(\bar{D}) := A_{\bar{D}}$. Then we can see that A_D^0, A_D^θ take values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$:

$$A_D^0 = \frac{1}{2} h'_\theta, \qquad A_D^\theta = -\beta\left(\frac{\partial}{\partial z}\right) - (A_D^0)^2.$$

Further

$$A_D^{\theta\bar{\theta}} = -\frac{\partial A_{\bar{D}}^0}{\partial z} + [A_{\bar{D}}^\theta, A_D^0] + [A_D^\theta, A_{\bar{D}}^0]$$
$$A_D^{\bar{\theta}} = -A_{\bar{D}}^\theta - [A_{\bar{D}}^0, A_D^0]$$

according to (27); so $A_D^{\theta\bar{\theta}}$, $A_D^{\bar{\theta}}$ are also with values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$ (these equations hold for left Maurer–Cartan forms but we have of course analogous equations for right Maurer–Cartan forms). Finally we have proved that $\gamma(D)$ takes values in $\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}$, so we have solved (37) in $\Lambda^+G_\tau^{\mathbb{C}}$. This completes the proof of the surjectivity (see [8]). For the characterization of the fibres it is the same proof as in [8]. ■

Let $\Phi\colon \mathbb{R}^{2|2} \to G/H$ be superharmonic with holomorphic potential $\mu \in \mathcal{SP}$, i.e. $\Phi = [\mathcal{F}_\mu]$ where $g = \mathcal{F}_\mu h$ and $g^{-1} \cdot \mathrm{d}g = \mu$, $i^*g(0) = 1$. Since g is holomorphic then by using (13), we can see that $g_0 = i^*g\colon \mathbb{R}^2 \to \Lambda G_\tau^{\mathbb{C}}$ is holomorphic:

$$\partial_{\bar{z}} g_0 = 0.$$

Furthermore, as in [8], let us consider the canonical map $\mathbf{det}\colon \Lambda G_\tau^{\mathbb{C}} \to Det^*$ (in [8], it is denoted by τ; see this reference for the definition of the map $\mathbf{det}$) and the set $|S| = (\mathbf{det} \circ g_0)^{-1}(0)$. Then according to [8], since g_0 is holomorphic and $\mathbf{det}\colon \Lambda G_\tau^{\mathbb{C}} \to Det^*$ is holomorphic, then $|S|$ is discrete. But, once more according to [8],

$$|S| = \{z \in \mathbb{R}^2 \mid g_0(z) \notin \text{ big cell}\}.$$

The result of this is that if we denote by S the discrete set $|S|$ endowed with the restriction to $|S|$ of the structural sheaf of $\mathbb{R}^{2|2}$, then the restriction of $g\colon \mathbb{R}^{2|2} \to \Lambda G_\tau^{\mathbb{C}}$ to the open submanifold of $\mathbb{R}^{2|2}$, $\mathbb{R}^{2|2} \setminus S$, takes values in the big cell (according to (6) since the big cell is an open set of $\Lambda G_\tau^{\mathbb{C}}$). Besides using the same arguments as in [8] we obtain that $S \subset \mathbb{R}^{2|2}$ depends only on the superharmonic map $\Phi\colon \mathbb{R}^{2|2} \to G/H$.

Theorem 10. *Let $\Phi\colon \mathbb{R}^{2|2} \to G/H$ be superharmonic and $S \subset \mathbb{R}^{2|2}$ as defined above. There exists a $\mathfrak{g}_1^{\mathbb{C}}$-valued odd holomorphic function η on $\mathbb{R}^{2|2} \setminus S$ such that*

$$\Phi = [\mathcal{F}_\mu]$$

on $\mathbb{R}^{2|2} \setminus S$, where

$$\mu = {I_{(D,\bar{D})}}^{-1}(\lambda^{-1}\eta, 0) = -\lambda^{-2}(\mathrm{d}z + (\mathrm{d}\theta)\theta)\eta^2 + \lambda^{-1}\mathrm{d}\theta\,\eta.$$

Proof. It is the same proof as in [8]. ■

6. The Weierstrass representation in terms of component fields

Let us consider a map $f\colon \mathbb{R}^{2|2} \to \mathbb{C}^n$; then by using (13), f is holomorphic if and only if $f = u + \theta\psi$ with u, ψ holomorphic on $\mathbb{R}^2$.

Further, according to the definition of a holomorphic potential, we can identify $\mathcal{SP}$ with the set of odd holomorphic maps $\mu(D)\colon \mathbb{R}^{2|2} \to \Lambda_{-1,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$. Such a map is written as

$$\mu(D) = \mu_D^0 + \theta\mu_D^\theta$$

where μ_D^0, μ_D^θ are holomorphic maps from $\mathbb{R}^2$ into $\Lambda_{-1,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$, μ_D^0 being odd and μ_D^θ being even. Now, let us embed $G^{\mathbb{C}}$ in $\mathrm{GL}_m(\mathbb{C})$ so that we can work in the vector space $\mathfrak{M}_m(\mathbb{C})$. Then the holomorphic map $g\colon \mathbb{R}^{2|2} \to \Lambda G_\tau^{\mathbb{C}}$ which integrates

$$g^{-1}Dg = \mu(D), \qquad i^*g(0) = 1$$

is the holomorphic map $g = g_0 + \theta g_\theta$ such that the holomorphic maps (g_0, g_θ) are solution of

$$g_0^{-1}\frac{\partial g_0}{\partial z} = -\left(\mu_D^\theta + (\mu_D^0)^2\right)$$
$$g_0^{-1}g_\theta = \mu_D^0.$$

Hence g_0 is the holomorphic map which comes from the (even) holomorphic potential $-(\mu_D^\theta + (\mu_D^0)^2)\mathrm{d}z$ defined on $\mathbb{R}^2$ and with values in $\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$. So we can see that the terms on λ^{-2} of the potential which we got rid by working on $\mu(D)$ instead of μ reappear now when we make explicit the Weierstrass representation in terms of the component fields.

Remark also that (g_0, g_θ) are the component fields of g. Thus we see that the writing of a holomorphic map is the same for every embedding, and that the third component field is equal to zero. Hence we can write $g = g_0 + \theta g_\theta$ without embedding $G^{\mathbb{C}}$; it is at the same time the writing of g in $\Lambda G_\tau^{\mathbb{C}}$, in $\Lambda\mathfrak{M}_m(\mathbb{C})$ and for every other embedding in a vector space $\Lambda\mathbb{C}^N$ (with $G^{\mathbb{C}} \hookrightarrow \mathbb{C}^N$).

Consider, now, the decomposition $g = \mathcal{F}h$, and write

$$\mathcal{F} = U + \theta_1\Psi_1 + \theta_2\Psi_2 + \theta_1\theta_2 f$$
$$h = h_0 + \theta_1 h_1 + \theta_2 h_2 + \theta_1\theta_2 h_{12}$$

(these are writings in $\Lambda\mathfrak{M}_m(\mathbb{C})$). Besides we have $g = g_0 + (\theta_1 + i\theta_2)g_\theta$. Hence we obtain

$$\begin{cases} g_0 = Uh_0 \\ g_\theta = \Psi_1 h_0 + Uh_1 \\ ig_\theta = \Psi_2 h_0 + Uh_2 \\ 0 = Uh_{12} + fh_0 + \Psi_2 h_1 - \Psi_1 h_2. \end{cases} \tag{38}$$

Thus U is obtained by decomposing g_0 which comes from a holomorphic potential, $-(\mu_D^\theta + (\mu_D^0)^2)\mathrm{d}z$, defined on $\mathbb{R}^2$ and with values in $\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$. So $u = i^*\Phi$ is the image by the Weierstrass representation of this potential.

Then, multiplying the second and third equations of (38) by U^{-1} on the left and by h_0^{-1} on the right, and remembering that $\Lambda\mathfrak{g}_\tau^{\mathbb{C}} = \Lambda\mathfrak{g}_\tau \oplus \Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}}$, we obtain that

$$\mathrm{Ad}h_0(\mu_D^0) = U^{-1}\Psi_1 + h_1 h_0^{-1}$$
$$i\mathrm{Ad}h_0(\mu_D^0) = U^{-1}\Psi_2 + h_2 h_0^{-1}$$

are the decompositions of $\mathrm{Ad}h_0(\mu_D^0)$ and $i\mathrm{Ad}h_0(\mu_D^0)$, respectively, following the previous direct sum. In particular, we have

$$U^{-1}\Psi_1 = \left[\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda\mathfrak{g}_\tau} \tag{39}$$

$$U^{-1}\Psi_2 = \left[i\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda\mathfrak{g}_\tau}. \tag{40}$$

Finally, the third component fields f', h'_{12} of $\mathcal{F}$ (resp. h) are the orthogonal projections of f (resp. h_{12}) on $U \cdot (\Lambda\mathfrak{g}_\tau)$ (resp. $(\Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}})h_0$). So by multiplying the last equation of (38) as above and by projecting on $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ we obtain

$$\left[(U^{-1}\Psi_1)(h_2 h_0^{-1}) - (U^{-1}\Psi_2)(h_1 h_0^{-1})\right]_{\Lambda\mathfrak{g}_\tau^{\mathbb{C}}} = U^{-1}f' + h'_{12}h_0^{-1}. \tag{41}$$

This is once again the decomposition of the left hand side following the direct sum $\Lambda\mathfrak{g}_\tau^{\mathbb{C}} = \Lambda\mathfrak{g}_\tau \oplus \Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}}$. Let us make precise the orthogonal projection

$$[\cdot]_{\Lambda\mathfrak{g}_\tau^{\mathbb{C}}} \colon \Lambda\mathfrak{M}_m(\mathbb{C}) \to \Lambda\mathfrak{g}_\tau^{\mathbb{C}}.$$

To do this it is enough to make precise $[\cdot]_{\mathfrak{g}^{\mathbb{C}}} \colon \mathfrak{M}_m(\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$. Since $\mathfrak{g}$ is semi-simple we can consider the embedding

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{so}(\mathfrak{g}) \subset \mathrm{gl}(\mathfrak{g}).$$

Besides, in $\mathrm{gl}(\mathfrak{g})$, we have the orthogonal direct sum $\mathrm{gl}(\mathfrak{g}) = \mathrm{so}(\mathfrak{g}) \oplus \mathrm{Sym}(\mathfrak{g})$. Then for $a, b \in \mathrm{so}(\mathfrak{g})$ the decomposition of ab is

$$ab = \frac{1}{2}[a, b] + \frac{ab + ba}{2}.$$

In particular for $a, b \in \mathfrak{g}$ this decomposition is the decomposition of ab following the direct sum $\mathrm{gl}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$. So

$$[ab]_{\mathfrak{g}} = \frac{1}{2}[a, b]. \tag{42}$$

Now let us extend τ to $\mathrm{gl}(\mathfrak{g})$ by taking $\mathrm{Ad}\tau$ (it is an extension because $\tau \circ \mathrm{adX} \circ \tau^{-1} = \mathrm{ad}(\tau(X))$). Then by the uniqueness of the writing of $\mathcal{F} = U + \theta_1 \Psi_1 + \theta_2 \Psi_2 + \theta_1\theta_2 f$ in $\Lambda\mathrm{gl}(\mathfrak{g})$ and since $\Lambda\mathrm{gl}(\mathfrak{g})_\tau$ is a vector subspace of $\Lambda\mathrm{gl}(\mathfrak{g})$, which contains ΛG_τ, we conclude that the previous writing is also the writing of $\mathcal{F}$ in $\Lambda\mathrm{gl}(\mathfrak{g})_\tau$. So $U^{-1}f$ takes values in $\Lambda\mathrm{gl}(\mathfrak{g})_\tau$ (and in the same way $h_{12}h_0^{-1}$ is with values in $\Lambda\mathrm{gl}(\mathfrak{g}^{\mathbb{C}})_\tau$). So, as τ commutes with the projection $[\cdot]_{\mathfrak{g}^{\mathbb{C}}}$ (because τ preserves the scalar product), in (41) it is enough to project in $\Lambda\mathfrak{g}^{\mathbb{C}}$ (following the direct sum $\Lambda\mathrm{gl}(\mathfrak{g}^{\mathbb{C}}) = \Lambda\mathfrak{g}^{\mathbb{C}} + \Lambda(\mathfrak{g}^{\perp})^{\mathbb{C}}$) and then we automatically project in $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ (following the direct sum $\Lambda\mathrm{gl}(\mathfrak{g}^{\mathbb{C}})_\tau = \Lambda\mathfrak{g}_\tau^{\mathbb{C}} + \Lambda(\mathfrak{g}^{\perp})_\tau^{\mathbb{C}}$).

Thus returning to the left hand side of (41), this is written as

$$\begin{aligned}&\frac{1}{2}\left[(U^{-1}\Psi_1), (h_2h_0^{-1})\right] - \frac{1}{2}\left[(U^{-1}\Psi_2), (h_1h_0^{-1})\right]\\ &\quad= \frac{1}{2}\left[\left[\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda\mathfrak{g}_\tau}, \left[i\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}}\right] - \frac{1}{2}\left[\left[i\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda\mathfrak{g}_\tau}, \left[\mathrm{Ad}h_0(\mu_D^0)\right]_{\Lambda^+\mathfrak{g}_\tau^{\mathbb{C}}}\right]\end{aligned}$$

by using (42), (39) and (40). Finally $U^{-1}f'$ is obtained by projecting this expression on $\Lambda\mathfrak{g}_\tau$ following the direct sum $\Lambda\mathfrak{g}_\tau^{\mathbb{C}} = \Lambda\mathfrak{g}_\tau \oplus \Lambda_{\mathfrak{b}}^+\mathfrak{g}_\tau^{\mathbb{C}}$. If we want $U^{-1}f$ (which depends on the embedding) we can write

$$(U^{-1}\Psi_1)(h_2h_0^{-1}) - (U^{-1}\Psi_2)(h_1h_0^{-1}) = U^{-1}f + h_{12}h_0^{-1}$$

and this is the decomposition of the left hand side following the direct sum $\Lambda\mathrm{gl}(\mathfrak{g}^{\mathbb{C}}) = \Lambda\mathrm{gl}(\mathfrak{g}) \oplus \Lambda^+\mathrm{gl}(\mathfrak{g}^{\mathbb{C}})$ (and this is also the decomposition following $\Lambda\mathrm{gl}(\mathfrak{g}^{\mathbb{C}})_\tau = \Lambda\mathrm{gl}(\mathfrak{g})_\tau \oplus \Lambda^+\mathrm{gl}(\mathfrak{g}^{\mathbb{C}})_\tau$ because all terms of the equation are twisted). Lastly, the component fields of $\Phi = \pi \circ \mathcal{F}_1$ are given by $u = \pi(U)$, $\psi_i = \mathrm{d}\pi(U) \cdot \Psi_i$ and $F' = 0$. For example, in the case $M = S^n$, π is just the restriction to $\mathrm{SO}(n+1)$ of the linear map which, with a matrix, associates its last column.

7. Primitive and superprimitive maps with values in a 4-symmetric space

7.1. The classical case

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$, $\sigma\colon G \to G$ an order 4 automorphism with the fixed point subgroup $G^\sigma = G_0$, and the corresponding Lie algebra $\mathfrak{g}_0 = \mathfrak{g}^\sigma$. Then G/G_0 is a 4-symmetric space. The automorphism σ gives us an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}_4} \tilde{\mathfrak{g}}_k$$

where $\tilde{\mathfrak{g}}_k$ is the $\mathrm{e}^{ik\pi/2}$-eigenspace of σ. We have clearly $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0^{\mathbb{C}}$, $\overline{\tilde{\mathfrak{g}}_k} = \tilde{\mathfrak{g}}_{-k}$ and $[\tilde{\mathfrak{g}}_k, \tilde{\mathfrak{g}}_l] \subset \tilde{\mathfrak{g}}_{k+l}$. We define $\mathfrak{g}_2$, $\underline{\mathfrak{g}}_1$ and $\mathfrak{m}$ by

$$\tilde{\mathfrak{g}}_2 = \mathfrak{g}_2^{\mathbb{C}}, \qquad \underline{\mathfrak{g}}_1^{\mathbb{C}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_1 \quad \text{and} \quad \mathfrak{m}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}_4\setminus\{0\}} \tilde{\mathfrak{g}}_k,$$

and this is possible because $\overline{\tilde{\mathfrak{g}}_2} = \tilde{\mathfrak{g}}_2$ and $\overline{\tilde{\mathfrak{g}}_{-1}} = \tilde{\mathfrak{g}}_1$. Let us set $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1}$, $\mathfrak{g}_1 = \tilde{\mathfrak{g}}_1$, $\underline{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. Then

$$\mathfrak{g} = \underline{\mathfrak{g}}_0 \oplus \underline{\mathfrak{g}}_1$$

is the eigenspace decomposition of the involutive automorphism $\tau = \sigma^2$. This is also a Cartan decomposition of $\mathfrak{g}$. Let $H = G^\tau$; then $\mathrm{Lie}\, H = \underline{\mathfrak{g}}_0$ and G/H is a symmetric space. We use the Killing form of $\mathfrak{g}$ to endow $N = G/G_0$ and $M = G/H$ with a G-invariant metric. For the homogeneous space $N = G/G_0$ we have the following reductive decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m} \tag{43}$$

($\mathfrak{m}$ can be written $\mathfrak{m} = \underline{\mathfrak{g}}_1 \oplus \mathfrak{g}_2$) with $[\mathfrak{g}_0, \mathfrak{m}] \subset \mathfrak{m}$. As for the symmetric space G/H, we can identify the tangent bundle TN with the subbundle $[\mathfrak{m}]$ of the trivial bundle $N \times \mathfrak{g}$, with fibre $\mathrm{Ad}\, g(\mathfrak{m})$ over the point $x = g \cdot G_0 \in N$. For every $\mathrm{Ad}\, G_0$-invariant subspace $\mathfrak{l} \subset \mathfrak{g}^{\mathbb{C}}$, we define $[\mathfrak{l}]$ in the same way as $[\mathfrak{m}]$. Then we introduce:

Definition 3. $\phi\colon \mathbb{R}^2 \to G/G_0$ is primitive if $\frac{\partial \phi}{\partial z}$ takes values in $[\mathfrak{g}_{-1}]$. Equivalently, this means that for any lift U of ϕ, with values in G, $U^{-1}\frac{\partial U}{\partial z}$ takes values in $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$.

We denote by $\pi_H\colon G \to G/H$, $\pi_{G_0}\colon G \to G/G_0$ and $p\colon G/G_0 \to G/H$ the canonical projections. Let $\phi\colon \mathbb{R}^2 \to G/G_0$, with U a lift, $\phi = \pi_{G_0} \circ U$, and $\alpha = U^{-1} \cdot \mathrm{d}U$. For α, we will use the following decompositions:

$$\alpha = \alpha_0 + \alpha_{\mathfrak{m}} \tag{44}$$

$$\alpha = \underline{\alpha}_0 + \underline{\alpha}_1 \tag{45}$$

$$\alpha = \alpha_2 + \alpha_{-1} + \alpha_0 + \alpha_1 \tag{46}$$

$$\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}} \tag{47}$$

where $\alpha'_{\mathfrak{m}}$ is a $(1,0)$-form and $\alpha''_{\mathfrak{m}}$ its complex conjugate. Using the decomposition (43), we want to write the equation of harmonic maps $\phi\colon \mathbb{R}^2 \to G/G_0$ in terms of the Maurer–Cartan form α, in the same way as for harmonic maps $u\colon \mathbb{R}^2 \to G/H$. Then we obtain, by using the identification $TN \simeq [\mathfrak{m}]$ (and so writing the harmonic maps equation in the form $\left[\bar\partial(\mathrm{Ad}U\alpha'_{\mathfrak{m}})\right]_{[\mathfrak{m}]} = 0$),

$$\bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_0 \wedge \alpha'_{\mathfrak{m}}] + [\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0. \tag{48}$$

Then if $[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0$, we have the same equation as for harmonic maps into a symmetric space, and in the same way, we can check (see [3]) that the extended Maurer–Cartan form

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_0 + \lambda\alpha''_{\mathfrak{m}} \tag{49}$$

satisfies the zero-curvature equation

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

Conversely, if the extended Maurer–Cartan form satisfies the zero-curvature equation and $[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0$, then ϕ is harmonic (see [3]).

In particular if we suppose that ϕ is primitive then $\alpha'_{\mathfrak{m}}$ takes values in $\mathfrak{g}_{-1}$ whereas $\alpha''_{\mathfrak{m}}$ takes values in $\overline{\mathfrak{g}_{-1}} = \mathfrak{g}_1$, so $[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0$. Moreover let us project the Maurer–Cartan equation for α onto $\mathfrak{g}_{-1}$:

$$\mathrm{d}\alpha'_{\mathfrak{m}} + [\alpha''_0 \wedge \alpha'_{\mathfrak{m}}] = 0.$$

This is the harmonic map equation (48) since $[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0$. So a primitive map $\phi\colon \mathbb{R}^2 \to G/G_0$ is harmonic. Moreover since the extended Maurer–Cartan form satisfies the zero-curvature equation, we can find a harmonic extended lift $U_\lambda\colon \mathbb{R}^2 \to \Lambda G$ such that $U_\lambda^{-1} \cdot \mathrm{d}U_\lambda = \alpha_\lambda$. Then $\phi_\lambda = \pi_{G_0} \circ U_\lambda$ is harmonic. Besides since ϕ is primitive the decomposition

$$\alpha = \alpha'_{\mathfrak{m}} + \alpha_0 + \alpha''_{\mathfrak{m}} \tag{50}$$

is also the decomposition (46) because $\alpha'_{\mathfrak{m}} \in \mathfrak{g}_{-1}$ so $\alpha'_{\mathfrak{m}} = \alpha_{-1}$, $\alpha''_{\mathfrak{m}} = \alpha_1$, $\alpha_2 = 0$, and then α_λ is a $\Lambda\mathfrak{g}_\sigma$-valued 1-form. Furthermore, decompositions (44) and (45) are the same and so the decomposition (50) can be rewritten as

$$\alpha = \underline{\alpha}'_1 + \underline{\alpha}_0 + \underline{\alpha}''_1$$

and then we can consider that α is the Maurer–Cartan form associated with $u = \pi_H \circ U = p \circ \phi$ with the corresponding extended Maurer–Cartan form α_λ given by (49). Then we conclude that $u_\lambda = p \circ \phi_\lambda\colon \mathbb{R}^2 \to G/H$ is harmonic and U_λ is an extended lift for it. Moreover, α_λ is also a $\Lambda\mathfrak{g}_\tau$-valued 1-form and $(U_\lambda)\colon \mathbb{R}^2 \to \Lambda G_\tau$. So we can write that $u = \mathcal{W}(\mu) = [U]$, where $\mathcal{W}\colon \mathcal{P} \to \mathcal{H}$ is the Weierstrass representation:

$$\mathcal{W}\colon \mu \in \mathcal{P} \mapsto g \text{ holomorphic } \mapsto (U, h) \in C^\infty(\mathbb{R}^2, \Lambda G_\tau \times \Lambda^+_{\mathcal{B}} G^{\mathbb{C}}_\tau) \mapsto \pi_H \circ U_1 \in \mathcal{H}$$

between the holomorphic potentials (holomorphic 1-forms μ taking values in $\Lambda_{-1,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$) and the harmonic maps (such that $u(0) = H$; see [8]). However to obtain μ we must solve the following $\bar{\partial}$-problem (see [8]):

$$\bar{\partial} h \cdot h^{-1} = -(\alpha_0'' + \lambda\alpha_1),$$

and since α_λ takes values in $\Lambda\mathfrak{g}_\sigma$, this is a $\bar{\partial}$-problem with right hand side in $\Lambda^+\mathfrak{g}_\sigma^{\mathbb{C}}$, so we can find a solution $h\colon \mathbb{R}^2 \to \Lambda^+ G_\sigma^{\mathbb{C}}$, $h(0) = 1$. Then the holomorphic map $g = Uh$ (it is holomorphic because h is a solution of the $\bar{\partial}$-problem) takes values in $\Lambda G_\sigma^{\mathbb{C}}$ and so the potential $\mu = g^{-1} \cdot \mathrm{d}g$ takes values in $\Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$. Let us write $\mathcal{P}_\sigma$, the vector subspace of $\mathcal{P}$, of holomorphic potentials taking values in $\Lambda_{-1,\infty}\mathfrak{g}_\sigma^{\mathbb{C}} = \Lambda_{-1,\infty}\mathfrak{g}_\tau^{\mathbb{C}} \cap \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$. Then we have proved that for each primitive map $\phi\colon \mathbb{R}^2 \to G/G_0$ there exists $\mu \in \mathcal{P}_\sigma$ such that $\phi = \pi_{G_0} \circ U$ where $g = Uh$ and $g^{-1} \cdot \mathrm{d}g = \mu$. However, the decomposition $g = Uh$ is in the same way the decomposition

$$\Lambda G_\tau^{\mathbb{C}} \overset{\mathrm{dec}_\tau}{=} \Lambda G_\tau \cdot \Lambda_{\mathcal{B}}^+ G_\tau^{\mathbb{C}}$$

but also

$$\Lambda G_\sigma^{\mathbb{C}} \overset{\mathrm{dec}_\sigma}{=} \Lambda G_\sigma \cdot \Lambda_{\mathcal{B}_0}^+ G_\sigma^{\mathbb{C}}$$

because g takes values in $\Lambda G_\sigma^{\mathbb{C}}$ and because of the uniqueness of the decomposition. We can say that the decomposition dec_σ (considered as a diffeomorphism) is the restriction of dec_τ to $\Lambda G_\sigma^{\mathbb{C}}$.

Conversely, let us prove that for any $\mu \in \mathcal{P}_\sigma$, $\phi = \pi_{G_0} \circ U_\mu$ is primitive, so that we can conclude that the map

$$\mathcal{W}_\sigma : \mu \in \mathcal{P}_\sigma \mapsto g \mapsto (U, h) \mapsto \phi = \pi_{G_0} \circ U_1$$

is a surjection between $\mathcal{P}_\sigma$ and the primitive maps, i.e. that it is a Weierstrass representation for primitive maps. So suppose that $\mu \in \mathcal{P}_\sigma$. Then we integrate it: $\mu = g^{-1} \cdot \mathrm{d}g$, $g(0) = 1$ and we decompose $g = Uh$ following dec_σ. Since it is also the decomposition following dec_τ, then we know (Weierstrass representation $\mathcal{W}$ for the symmetric space G/H) that $\alpha_\lambda = U_\lambda^{-1} \cdot \mathrm{d}U_\lambda$ is in the form

$$\alpha_\lambda = \lambda^{-1}\underline{\alpha}_1' + \underline{\alpha}_0 + \lambda\underline{\alpha}_1''$$

but since α_λ is with values in $\Lambda\mathfrak{g}_\sigma$ (because U takes values in ΛG_σ) then $\underline{\alpha}_1' \in \mathfrak{g}_{-1}$, $\underline{\alpha}_0 \in \mathfrak{g}_0$, $\underline{\alpha}_1'' \in \mathfrak{g}_1$ so $\phi_\lambda = \pi_{G_0} \circ U_\lambda$ is primitive.

Hence we have proved the following:

Theorem 11. *We have a Weierstrass representation for primitive maps; more precisely the map*

$$\begin{array}{llllllll} \mathcal{W}_\sigma : & \mathcal{P}_\sigma & \xrightarrow{\mathrm{int}} & \mathrm{H}(\mathbb{C}, \Lambda G_\sigma^{\mathbb{C}}) & \xrightarrow{\mathrm{dec}_\sigma} & C^\infty(\mathbb{R}^2, \Lambda G_\sigma \times \Lambda_{\mathcal{B}_0}^+ G_\sigma^{\mathbb{C}}) & \longrightarrow & \mathrm{Prim}(G/G_0) \\ & \mu & \longmapsto & g & \longmapsto & (U, h) & \longmapsto & \phi = \pi_{G_0} \circ U_1 \end{array}$$

is surjective. $\mathrm{H}(\mathbb{C}, \Lambda G_\sigma^{\mathbb{C}})$ *is the set of holomorphic maps from* $\mathbb{C}$ *to* $\Lambda G_\sigma^{\mathbb{C}}$*, and* $\mathrm{Prim}(G/G_0)$ *is the set of primitive maps* $\phi\colon \mathbb{R}^2 \to G/G_0$ *so that* $\phi(0) = G_0$*. We can say that* $\mathcal{W}_\sigma$ *is the restriction of the Weierstrass representation* $\mathcal{W}$ *for harmonic maps into* G/H *to the subspace* $\mathcal{P}_\sigma$*. More precisely, we have the following commutative diagram:*

$$\begin{array}{ccccccc} \mathcal{P} & \xrightarrow{\mathrm{int}} & \mathrm{H}(\mathbb{C}, \Lambda G_\tau^{\mathbb{C}}) & \xrightarrow{\mathrm{dec}_\tau} & C^\infty(\mathbb{R}^2, \Lambda G_\tau \times \Lambda_{\mathcal{B}}^+ G_\tau^{\mathbb{C}}) & \xrightarrow{[\pi_H]} & \mathcal{H} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle [p]} \\ \mathcal{P}_\sigma & \xrightarrow{\mathrm{int}} & \mathrm{H}(\mathbb{C}, \Lambda G_\sigma^{\mathbb{C}}) & \xrightarrow{\mathrm{dec}_\sigma} & C^\infty(\mathbb{R}^2, \Lambda G_\sigma \times \Lambda_{\mathcal{B}_0}^+ G_\sigma^{\mathbb{C}}) & \xrightarrow{[\pi_{G_0}]} & \mathrm{Prim}(G/G_0) \end{array}$$

where $[\pi_H](U, h) = \pi_H \circ U_1$, $[p](\phi) = p \circ \phi$*. In particular the image under* $\mathcal{W}$ *of* $\mathcal{P}_\sigma$ *is the subset of* $\mathcal{H}$: $\{u = p \circ \phi,\ \phi\ \text{primitive}\}$.

7.2. The supersymmetric case

Definition 4. A superfield $\tilde{\Phi}\colon \mathbb{R}^{2|2} \to G/G_0$ is primitive if $D\tilde{\Phi}$ takes values in $[\mathfrak{g}_{-1}]$. Equivalently, this means that for any lift $\mathcal{F}$ of $\tilde{\Phi}$, with values in G, $U^{-1} \cdot DU$ takes values in $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$.

By proceeding as above and using the methods that we developed in the previous sections to work in the superspace, we obtain the following two theorems:

Theorem 12. *Let $\tilde{\Phi}\colon \mathbb{R}^{2|2} \to G/G_0$, a superfield, $\mathcal{F}\colon \mathbb{R}^{2|2} \to G$, a lift, and $\alpha = \mathcal{F}^{-1}\cdot \mathrm{d}\mathcal{F}$, its Maurer–Cartan form. Then $\tilde{\Phi}$ is superharmonic if and only if*

$$\bar{D}\alpha_{\mathfrak{m}}(D) + [\alpha_0(\bar{D}), \alpha_{\mathfrak{m}}(D)] + [\alpha_{\mathfrak{m}}(\bar{D}), \alpha_{\mathfrak{m}}(D)]_{\mathfrak{m}} = 0.$$

Further if $[\alpha_{\mathfrak{m}}(\bar{D}), \alpha_{\mathfrak{m}}(D)]_{\mathfrak{m}} = 0$, then the pair $(\alpha_0(D) + \lambda^{-1}\alpha_{\mathfrak{m}}(D), \alpha_0(\bar{D}) + \lambda\alpha_{\mathfrak{m}}(\bar{D}))$ satisfies the zero-curvature equation (25), and so yields by $I^{-1}_{(D,\bar{D})}$ an extended Maurer–Cartan form α_λ. In particular, if $\tilde{\Phi}$ is superprimitive then $[\alpha_{\mathfrak{m}}(\bar{D}), \alpha_{\mathfrak{m}}(D)]_{\mathfrak{m}} = 0$, $\tilde{\Phi}$ is superharmonic and $\Phi = p \circ \tilde{\Phi}\colon \mathbb{R}^{2|2} \to G/H$ is superharmonic.

Theorem 13. *We have a Weierstrass representation for superprimitive maps; more precisely with obvious notation (according to the foregoing)*

$$\begin{array}{llllllll} \mathcal{SW}_\sigma : & \mathcal{SP}_\sigma & \xrightarrow{\text{int}} & \mathrm{H}(\mathbb{R}^{2|2}, \Lambda G_\sigma^{\mathbb{C}}) & \xrightarrow{\mathrm{dec}_\sigma} & C^\infty(\mathbb{R}^{2|2}, \Lambda G_\sigma \times \Lambda^+_{\mathcal{B}_0} G_\sigma^{\mathbb{C}}) & \longrightarrow & \mathrm{SPrim}(G/G_0) \\ & \mu & \longmapsto & g & \longmapsto & (\mathcal{F}, h) & \longmapsto & \tilde{\Phi} = \pi_{G_0} \circ \mathcal{F}_1 \end{array}$$

is surjective. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{SP} \arrow[r, "\text{int}"] & \mathrm{H}(\mathbb{R}^{2|2}, \Lambda G_\tau^{\mathbb{C}}) \arrow[r, "\mathrm{dec}_\tau"] & C^\infty(\mathbb{R}^{2|2}, \Lambda G_\tau \times \Lambda^+_{\mathcal{B}} G_\tau^{\mathbb{C}}) \arrow[r, "{[\pi_H]}"] & \mathcal{SH} \\ \mathcal{SP}_\sigma \arrow[u] \arrow[r, "\text{int}"] & \mathrm{H}(\mathbb{R}^{2|2}, \Lambda G_\sigma^{\mathbb{C}}) \arrow[u] \arrow[r, "\mathrm{dec}_\sigma"] & C^\infty(\mathbb{R}^{2|2}, \Lambda G_\sigma \times \Lambda^+_{\mathcal{B}_0} G_\sigma^{\mathbb{C}}) \arrow[u] \arrow[r, "{[\pi_{G_0}]}"] & \mathrm{SPrim}(G/G_0) \arrow[u, "{[p]}"'] \end{tikzcd}$$

In particular the image by $\mathcal{SW}$ of $\mathcal{SP}_\sigma$ is the subset of $\mathcal{SH}$:

$$\{\Phi = p \circ \tilde{\Phi},\ \tilde{\Phi} \text{ primitive}\}.$$

Here, the holomorphic potentials of $\mathcal{SP}_\sigma$ take values in $\Lambda_{-2,\infty}\mathfrak{g}_\sigma^{\mathbb{C}}$ and the corresponding extended Maurer–Cartan form is in the form (35) but with values in $\Lambda\mathfrak{g}_\sigma \subset \Lambda\mathfrak{g}_\tau$ (for example, in (35) $\alpha_1(D)$ takes values in $\mathfrak{g}_{-1}$ so $\alpha_1(D)^2$ takes values in $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_2^{\mathbb{C}}$).

8. The second elliptic integrable system associated with a 4-symmetric space

We give ourselves the same ingredients and notation as at the beginning of Section 7.1. Then let us recall what a second elliptic system is according to Terng (see [25]).

Definition 5. The second (G, σ)-system is the equation for $(u_0, u_1, u_2)\colon \mathbb{C} \to \bigoplus_{j=0}^{2} \tilde{\mathfrak{g}}_{-j}$,

$$\begin{cases} \partial_{\bar{z}} u_2 + [\bar{u}_0, u_2] = 0 \\ \partial_{\bar{z}} u_1 + [\bar{u}_0, u_1] + [\bar{u}_1, u_2] = 0 \\ -\partial_{\bar{z}} u_0 + \partial_z \bar{u}_0 + [u_0, \bar{u}_0] + [u_1, \bar{u}_1] + [u_2, \bar{u}_2] = 0. \end{cases} \tag{51}$$

It is equivalent to say that the 1-form

$$\alpha_\lambda = \sum_{i=0}^{2} \lambda^{-i} u_i \mathrm{d}z + \lambda^i \bar{u}_i \mathrm{d}\bar{z} = \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'' + \lambda^2\alpha_2'' \tag{52}$$

satisfies the zero-curvature equation:

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

The first example of second elliptic system was given by Hélein and Romon (see [15–17]): they showed that the equations for Hamiltonian stationary surfaces in four-dimensional Hermitian symmetric spaces form the second elliptic system associated with certain 4-symmetric spaces. Then we generalized the case of $\mathbb{R}^4 = \mathbb{H}$ (see [15]) in the space $\mathbb{R}^8 = \mathbb{O}$ (with $G = \mathrm{Spin}(7) \ltimes \mathbb{O}$, $\sigma = int_{(-L_e,0)}$, where int_g is the conjugation with g, $e \in S(\mathrm{Im}\,\mathbb{O})$, and L_e is the left multiplication by e; see [19,10,11]): there exists a family $(\mathcal{S}_I)$ of sets of surfaces in $\mathbb{O}$, indexed by $I \subsetneqq \{1, \dots, 7\}$, called the ρ-harmonic ω_I-isotropic surfaces, such that $\mathcal{S}_I \subset \mathcal{S}_J$ if $J \subset I$, and whose equations are the second elliptic (G, σ)-system (see [19]). We think that our result can be generalized to $\mathbb{OP}^1$, $\mathbb{OP}^2$ or more simply to $\mathbb{HP}^1$.

For any second elliptic system associated with a 4-symmetric space, we can use the method of [8] to construct a Weierstrass representation, defined on $\mathcal{P}^2_\sigma$, the vector space of $\Lambda_{-2,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$-valued holomorphic 1-forms on $\mathbb{C}$ (see [15,17]):

$$\mathcal{W}^2_\sigma : \mathcal{P}^2_\sigma \xrightarrow{\text{int}} \mathrm{H}(\mathbb{C}, \Lambda G^{\mathbb{C}}_\sigma) \xrightarrow{\mathrm{dec}_\sigma} C^\infty(\mathbb{R}^2, \Lambda G_\sigma \times \Lambda^+_{\mathcal{B}_0} G^{\mathbb{C}}_\sigma) \xrightarrow{[\pi]} \mathcal{S}$$

where $\mathcal{S}$ is the set of geometric maps whose equations correspond to the second elliptic system, and $[\pi](U, h) = \pi \circ U_1$. π can be π_{G_0} as well as π_H. For example in the case of Hamiltonian stationary surfaces in a Hermitian symmetric space G/H, we must take π_H (see [17]). Moreover if we consider the solution $u = \mathcal{W}^2_\sigma(\mu) = \pi_H \circ U_1$, then in this case $\phi = \pi_{G_0} \circ U_1$ can be identified with the map $(u, \mathrm{e}^{\mathrm{i}\beta})$ where β is a Lagrangian angle function of u ($G/G_0 = G \times_H (H/G_0)$ is the principal $U(1)$-bundle $U(G/H)/SU(2)$). If we restrict $\mathcal{W}^2_\sigma$ to $\mathcal{P}_\sigma$, we obtain $\mathcal{W}_\sigma$, the Weierstrass representation of primitive maps, whose image is the set of special Lagrangian surfaces of G/H (by identifying u and $\phi = (u, 1)$).

Now, we are going to give another example of second elliptic system in the even part of a super-Lie algebra. According to the previous section, a superprimitive map $\tilde{\Phi} : \mathbb{R}^{2|2} \to G/G_0$ leads to a extended lift $\mathcal{F} : \mathbb{R}^{2|2} \to \Lambda G_\sigma$. Let us consider $U = i^*\mathcal{F} : \mathbb{R}^2 \to \Lambda G_\sigma$; then according to Section 6, U is obtained from an (even) holomorphic potential, $-(\mu^\theta_D + (\mu^0_D)^2)\mathrm{d}z$, which is defined in $\mathbb{R}^2$ and with values in $\Lambda_{-2,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$. This is a $\Lambda_{-2,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$-valued holomorphic 1-form on $\mathbb{R}^2$. In concrete terms, if we consider that we work with the category of supermanifolds (sets of parameters B; see the introduction) $\{\mathbb{R}^{0|L}, L \in \mathbb{N}\}$, i.e. that we work with G^∞ functions defined on $B^{2|2}_L$ (see [24]), then this is a $(\Lambda_{-2,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma \otimes B^0_L)$-valued holomorphic 1-form on $\mathbb{R}^2$. In other words U comes from a holomorphic potential which is in $\mathcal{P}^2_\sigma \otimes B^0_L$. So $u = \pi_H \circ U_1 : \mathbb{R}^2 \to G/H$ like $\phi = \pi_{G_0} \circ U_1 : \mathbb{R}^2 \to G/G_0$ corresponds to a solution of the second elliptic system (51) in the Lie algebra $\mathfrak{g} \otimes B^0_L$ (i.e. u_i takes values in $\tilde{\mathfrak{g}}_{-i} \otimes B^0_L$). However that does not give us a supersymmetric interpretation of all second elliptic systems (51) in the Lie algebra $\mathfrak{g}$ in terms of superprimitive maps. Indeed, first the coefficient on λ^{-2} of the previous potential does not have a body term: it is the square of an odd element so it does not have terms on $1 = \eta^\varnothing$ (we set $B_L = \mathbb{R}[\eta_1, \dots \eta_L]$). Second, this coefficient takes values in $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ which can be $\subsetneqq \mathfrak{g}^{\mathbb{C}}_2$.

In conclusion, the restrictions to $\mathbb{R}^2$ of superprimitive maps $\tilde{\Phi} : \mathbb{R}^{2|2} \to G/G_0$ correspond to particular solutions of the second elliptic system (51) in the Lie algebra $\mathfrak{g} \otimes B^0_L$: those which come by $\mathcal{W}^2_\sigma$, from potentials in the form $\hat{\mu} = -(\mu^\theta_D + (\mu^0_D)^2)\mathrm{d}z$, with $\mu \in \mathcal{SP}_\sigma$.

Besides, for each 4-symmetric space (G, σ), this gives us a geometrical interpretation of certain solutions of the second elliptic system (51) in $\mathfrak{g} \otimes B^0_L$. Hence this confirms our conjecture that there exist geometrical problems in $\mathbb{HP}^1$, $\mathbb{OP}^1$ and $\mathbb{OP}^2$, analogous to the ρ-harmonic surfaces in $\mathbb{O}$ [19], whose equations are respectively the second elliptic problems in the 4-symmetric spaces equal to the twistor spaces of $\mathbb{HP}^1 = \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$, $\mathbb{OP}^1 = \mathrm{Spin}(9)/\mathrm{Spin}(8)$ and $\mathbb{OP}^2 = F_4/\mathrm{Spin}(9)$.

Let us give an example by considering the case of the 4-symmetric space $\mathrm{SU}(3)/\mathrm{SU}(2)$ (used by Hélein and Romon for their study of Hamiltonian stationary surfaces in $\mathbb{CP}^2 = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$).

Theorem 14. *Consider the case of the 4-symmetric space* $\mathrm{SU}(3)/\mathrm{SU}(2)$ *(*$H = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$*). Then an immersion* $u : \mathbb{R}^2 \to \mathbb{CP}^2(\mathbb{R}^{0|L})$ *from* $\mathbb{R}^2$ *to the* G^∞ *manifold over* B_L *of* $\mathbb{R}^{0|L}$*-points of* $\mathbb{CP}^2$ *(morphisms from* $\mathbb{R}^{0|L}$ *to* $\mathbb{CP}^2$*) is the restriction to* $\mathbb{R}^2$ *of a superprimitive map*

$$\tilde{\Phi} : \mathbb{R}^{2|2} \to \mathrm{SU}(3)/\mathrm{SU}(2)$$

(i.e. $u = p \circ \tilde{\Phi} \circ i$*) if and only if* u *is a Lagrangian conformal immersion whose Lagrangian angle* β *satisfies*

$$\frac{\partial \beta}{\partial z} = ab \tag{53}$$

where $a, b\colon \mathbb{R}^2 \to \mathbb{C}[\eta_1, \ldots, \eta_L]$ *are odd holomorphic functions. In this case, we have* $\phi = i^*\tilde{\Phi} = (u, \mathrm{e}^{\mathrm{i}\beta})$.

Proof. Suppose that u is the restriction to $\mathbb{R}^2$ of a superprimitive map $\tilde{\Phi}$; then u is the image by the Weierstrass representation $\mathcal{W}^2_\sigma$ of the holomorphic potential $\hat{\mu} = -(\mu^\theta_D + (\mu^0_D)^2)\mathrm{d}z$ with $\mu \in \mathcal{SP}_\sigma$. Thus u is a Lagrangian conformal immersion. Let us set

$$\mu_D = \lambda^{-1}(A^0 + \theta A^\theta) + \sum_{k\geq 0} \lambda^k \left((\mu^0_D)_k + \theta(\mu^\theta_D)_k\right),$$

where A^0, A^θ takes values in $\mathfrak{g}_{-1}$; then

$$\hat{\mu} = -\lambda^{-2}(A^0)^2\mathrm{d}z + \sum_{k\geq -1} \lambda^k \hat{\mu}_k.$$

Next, since A^0 is in $\mathfrak{g}_{-1} \otimes B^1_L$, we can write (see [17])

$$A^0 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -ib & ia & 0 \end{pmatrix} \tag{54}$$

and thus

$$\hat{\mu}_{-2} = iab \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathrm{d}z = 3abY\mathrm{d}z$$

where $Y = \frac{i}{3}\mathrm{Diag}(1, 1, -2)$. If we denote by $\hat{\alpha}_\lambda = U^{-1}\mathrm{d}U = i^*\alpha_\lambda$ the extended Maurer–Cartan form associated with u, then u is an immersion if and only if $\hat{\alpha}_{-1}$ does not vanish. Besides since $\mathfrak{g}^{\mathbb{C}}_2 = \mathbb{C}Y$, one can easily see that

$$\hat{\alpha}'_2 = \hat{\mu}_{-2}$$

(because $[\mathfrak{g}_0, \mathfrak{g}_2] = 0$). Moreover we have (see [17])

$$\frac{\mathrm{d}\beta}{2}Y = \hat{\alpha}_2$$

and so finally

$$\frac{\partial\beta}{\partial z} = 6ab.$$

Conversely, suppose that u is a Lagrangian conformal immersion which satisfies (53). Then we have $\triangle\beta = 0$ since a, b are holomorphic by hypothesis. So we can write $u = \mathcal{W}^2_\sigma(\hat{\mu})$ with $\hat{\mu} \in \mathcal{P}^2_\sigma \otimes B^0_L$. Let us take for $\hat{\mu}$ a meromorphic potential (see [17])

$$\hat{\mu} = \lambda^2\hat{\mu}_{-2} + \lambda^{-1}\hat{\mu}_{-1}.$$

Then according to (53) we have $\hat{\mu}_{-2} = -(A^0)^2\mathrm{d}z$ with A^0 in the same form as in (54). Thus if we set $\mu_D = \lambda^{-1}(A^0 - \theta\hat{\mu}_{-1}(\frac{\partial}{\partial z}))$, then μ_D is an odd meromorphic map from $\mathbb{R}^{2|2}$ to $\Lambda_{-1,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$ and we have $\hat{\mu} = -(\mu^\theta_D + (\mu^0_D)^2)\mathrm{d}z$, so $u = p \circ \tilde{\Phi} \circ i$ with $\tilde{\Phi} = \mathcal{SW}_\sigma(I^{-1}_{(D,\bar{D})}(\mu_D, 0))$. ■

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